AMSI VACATION RESEARCH SCHOLARSHIPS 2018–19

# The Shape of a Drum

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Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

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Department of Education and Training

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Consider a bounded planar domain $\Omega \subset \mathbb{R}^2$ that vibrates according to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u; \;\; u|_{\partial\Omega} = 0$$

(where $\Delta = \nabla^2$ is the Laplacian). In solving this equation one arrives at a simpler pde:

$$\Delta u = -\lambda u; \;\; u|_{\partial\Omega} = 0.$$

From solving this problem, one can work out the frequencies that this domain (drum) vibrates at. These frequencies are determined by the eigenvalues $\lambda$.

# The general problem

Mark Kac considered the inverse problem:

**Problem**

If we know the spectrum of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ for the Dirichlet eigenproblem

$$\Delta u = -\lambda u; \;\; u|_{\partial\Omega} = 0 \tag{1}$$

on a bounded domain $\Omega$ in the plane, can we determine what $\Omega$ looks like exactly?

That is, can one hear the shape of a drum?

## The general answer

It turns out the answer is no in general, with a counterexample found by Gordon, Webb and Wolpert given below:

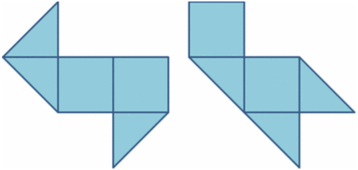

Figure: Identical sounding drums

They constructed this example by an abstract group theoretic method called the Sunada method.

## But not all hope is lost!

We can recover some geometric information from the spectrum including the area of the domain, the perimeter of the domain, and the presence of corners.

Such quantities are examples of so called *spectral invariants*.

One of the primary tools used to study this problem is the heat trace:

$$h(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}.$$

## The Heat Trace

To justify the name 'heat trace' we begin with the heat equation:

$$\begin{cases} (\partial_t - \Delta)u = 0, & \\ u(x,0) = f(x) & x \in \Omega, \\ u(x,t) = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

where $f$ is some initial heat distribution over $\Omega$.
Using separation of variables one finds that a solution to (2) is $u(x,t) = e^{-\lambda_k t}\phi_k(x)$ where $k \in \mathbb{N}$, and $\phi_k$ satisfies

$$\Delta\phi_k = -\lambda_k\phi_k; \ \phi_k|_{\partial\Omega} = 0.$$

Moreover, $\{\phi_k\}_{k=1}^{\infty}$ forms an orthonormal basis for $L^2(\Omega)$.

## The Heat Trace

Thus we can write $f = \sum_{k=1}^{\infty} \langle f, \phi_k \rangle \phi_k$, and therefore we have

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \langle f, \phi_k \rangle e^{-\lambda_k t} \phi_k(x) \\ &= \sum_{k=1}^{\infty} \left( \int_\Omega f(y)\phi_k(y)\, dy \right) e^{-\lambda_k t} \phi_k(x) \\ &= \int_\Omega \left( \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x)\phi_k(y) \right) f(y)\, dy \\ &= \int_\Omega H(x,y,t) f(y)\, dy, \end{aligned}$$

where $H(x,y,t)$ is the heat kernel. Note: $\langle f, g \rangle = \int_\Omega f(x)g(x)\, dx$.

# The Heat Trace

In the finite-dimensional case, to compute the trace of a matrix we sum the diagonal. Now since we are in infinite dimensions, the appropriate thing to do is integrate the heat kernel over the diagonal. That is, we set $x = y$. Thus, the heat trace is given by

$$h(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_{\Omega} H(x, x, t)\, dx.$$

$$\left( H(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y) \right)$$

## An important result

The power of the heat trace comes from studying its short time asymptotics.

**Theorem**

For a bounded planar domain $\Omega$ with piecewise smooth boundary and a set of corners $C = \{p_j\}_{j=1}^n$ with interior angles $\{\theta_j\}_{j=1}^n$, the heat trace satisfies

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{12}\int_{\partial\Omega\setminus C} k\, ds + \frac{1}{24}\sum_{j=1}^{n}\left(\frac{\pi}{\theta_j} - \frac{\theta_j}{\pi}\right)$$

as $t \to 0$.

## The heat trace for a rectangle

Consider a rectangle $\Omega$ of side lengths $a$ and $b$ in the plane. Upon solving the Dirichlet eigenproblem (1), we find that the eigenfunctions and eigenvalues are given by

$$\phi_{mn}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

and $\lambda_{mn} = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2$ where $n, m \in \mathbb{Z}^+$.

# The heat trace for a rectangle

The heat trace is given by

$$h(t) = \sum_{m,n} e^{-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2 t} = \left(\sum_{n=1}^{\infty} e^{-\frac{n^2}{a^2}\pi^2 t}\right)\left(\sum_{m=1}^{\infty} e^{-\frac{m^2}{b^2}\pi^2 t}\right).$$

This is a rather complicated looking sum to deal with, so we will use the Poisson summation formula as a tool to deal with it.

# The Poisson summation formula

**Lemma (Poisson summation formula)**
For a suitably nice function $f$, the following holds:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$$

where $\hat{f}$ is the Fourier transform of $f$.

This formula allows us to deal with some sums that would otherwise be tricky.

# The heat trace for a rectangle

Using the Poisson summation formula we find that

**Lemma**

For all $a > 0$, $k \in \mathbb{Z}^+$ we have

$$\sum_{n \in \mathbb{Z}^+} e^{-\frac{\pi^2}{a^2} n^2 t} = -\frac{1}{2} + \frac{a}{2\sqrt{\pi t}} + o(t^k)$$

as $t \to 0^+$.

## The heat trace for a rectangle

Using our lemma we have

$$
\begin{aligned}
h(t) &= \left(\sum_{n=1}^{\infty} e^{-\frac{n^2}{a^2}\pi^2 t}\right)\left(\sum_{m=1}^{\infty} e^{-\frac{m^2}{b^2}\pi^2 t}\right) \\
&= \left(-\frac{1}{2} + \frac{a}{2\sqrt{\pi t}} + o(t^k)\right)\left(-\frac{1}{2} + \frac{b}{2\sqrt{\pi t}} + o(t^k)\right) \\
&= \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + o(t^{k-1/2})
\end{aligned}
$$

as $t \to 0^+$ for all $k \in \mathbb{Z}^+$.

## The heat trace for a rectangle

**Theorem (Heat trace for rectangle)**
For all $k \in \mathbb{Z}^+$, the heat trace for the rectangle $\Omega$ above is given by

$$h(t) = \frac{ab}{4\pi t} - \frac{2(a+b)}{8\sqrt{\pi t}} + \frac{1}{4} + o(t^{k-1/2})$$

as $t \to 0^+$.

As a reminder, the general heat trace expression is the following:

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{12}\int_{\partial\Omega\setminus C} k\, ds + \frac{1}{24}\sum_{j=1}^{n}\left(\frac{\pi}{\theta_j} - \frac{\theta_j}{\pi}\right).$$

## Comments

- As we can see from the heat trace computation above, the area of the rectangle, $ab$, and the perimeter, $2(a+b)$, are both clearly given in the asymptotic expansion. This is what we expect from the general heat trace expansion.
- Upon plugging in $k=0$ for the geodesic curvature since the rectangle sides are straight lines, and the angles of the corners all being $\pi/2$ we see that the constant term $1/4$ matches perfectly with the general heat trace expansion.

# The Wave Trace

It turns out that the heat trace isn't always sufficient and so one often uses the 'wave trace' defined by:

$$w(t) := \sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k}t}.$$

While the wave trace doesn't always exist as a function it may be studied as a 'tempered distribution'. One can analyse a set called the singular support

$$\text{singsupp } w$$

which is essentially the set of points for which $w$ is not a smooth function. Since the singular support is determined by the wave trace, it is a spectral invariant.

# The Length Spectrum

Consider a billiard table. Suppose we set a billiard ball in motion along this table and consider the set of paths for which the ball returns to its starting point and continues along the same path again. We call these paths the set of *closed geodesics*.

The *length spectrum* is the set of lengths of these closed geodesics counted with multiplicities. Given suitable hypotheses, the length spectrum is equal to the singular support, and so the length spectrum is another spectral invariant in this case.

## Some Open Problems

All the known counterexamples to the general problem are non-convex polygons. So naturally the following are still open problems:

- Can one hear the shape of a convex polygon?
- Can one hear the shape of a domain with smooth boundary?