A LINK BETWEEN QUOTIENTS IN SYMPLECTIC AND ALGEBRAIC GEOMETRY

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> *"This is the real secret of life – to be completely engaged with what you are doing in the here and now. And instead of calling it work, realize it is play."*
>
> – Alan Watts

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Following a rocky start to my Honours year in late 2019, due to suffering a personal setback, I eagerly began work on my thesis topic. In 2020, the social isolation due to the pandemic proved rather tricky to deal with, and I realised that I had taken for granted the value of passing conversations with fellow classmates on campus. Particularly challenging for me was adjusting to working from home, a place in which I am more vulnerable to bad habits. Nevertheless, this year has taught me a great deal about how to live.

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Abstract

In their 1982 paper 'Geometric quantization and multiplicities of group representations', Guillemin and Sternberg [GS82] discovered a topological connection between the symplectic quotient and the GIT quotient both constructed from an algebraic group acting on a projective variety. We investigate quotients in symplectic and algebraic geometry, and then showcase Guillemin and Sternberg's work, concluding with a proof of the topological link between them.

Introduction

Symplectic geometry is an area of mathematics that studies symplectic manifolds, which are geometric objects that generalise the phase space from Hamilton's formulation of Newtonian mechanics, formulated in 1833. Developing separately but alongside symplectic geometry, was the subject of algebraic geometry. Algebraic geometry is a branch of mathematics that uses algebraic tools to study geometric questions about objects such as algebraic varieties, which are spaces cut out by polynomial equations. Classically, algebraic geometry has its origins with the study of polynomial equations in many variables.

For many decades, the two geometric theories evolved in parallel, but in the 1960s, they began to converge. A particular area within their convergence was found by investigating group actions on the geometric objects in symplectic geometry and algebraic geometry, which may be thought of as the symmetries of the geometric objects (e.g. rotating a sphere does not change the way it looks). In symplectic geometry, one looks at so-called *Hamiltonian* group actions on symplectic manifolds, while in algebraic geometry, one considers so-called *reductive* group actions on algebraic varieties.

Amongst the main questions mathematicians asked in their investigations of group actions in symplectic and algebraic geometry, was how to construct the 'right' definition of a group quotient-object (or simply, quotient) in each of the geometric theories. Mathematicians wanted such a group quotient-object to encode as much information as possible about how symmetry simplifies the study of the geometric objects, while ensuring that the quotients themselves be geometric objects of the same type. In other words, within symplectic geometry, for example, quotienting a symplectic manifold by a suitable group action should result in another symplectic manifold.

The question of how to 'correctly' define a group quotient-object turned out to be surprisingly subtle in both symplectic geometry and algebraic geometry. For instance, in both geometric theories, the ordinary topological quotient (which, we remind the reader, is the set of orbits of points in a topological space with a given group action) does not, in general, suffice as the desired definition of group quotient-object.

In symplectic geometry, a suitable notion for a group quotient-object (which we refer to as a symplectic quotient) was not found until 1973 by Meyer [Mey73], and independently in 1974 by Marsden and Weinstein [MW74]. The resulting symplectic quotient is called the Marsden-Weinstein-Meyer reduction, or simply, the symplectic reduction. We explore the work of Marsden, Weinstein and Meyer in Chapter 2.

While, in algebraic geometry, a suitable notion for a group quotient-object was not found until 1965, when Mumford introduced his *Geometric Invariant Theory* (GIT) [MFK94]. Inspiration for Mumford's GIT came from Hilbert's largely unknown ideas in his 1893 paper on classical invariant theory, which is an old subject focused most on questions about invariants of binary forms. The main mathematical construction within GIT was the definition of the GIT quotient, given by Mumford. The GIT quotient turned out to be the 'right' notion of quotient, within algebraic geometry, that mathematicians had long been in search of. We investigate the subtlety of Mumford's GIT in Chapter 4.

Before we can look at geometric invariant theory in Chapter 4, we must first intro *classical* algebraic geometry in Chapter 3. We explain what we mean by classical algebraic geometry as follows. In the 1950s, Grothendieck introduced new and abstract machinery to algebraic geometry, such as his theory of schemes. Grothendieck's work revolutionised the subject and essentially separated it into classical algebraic geometry and modern algebraic geometry. In this thesis, we only discuss ideas from classical algebraic geometry and leave out the more advanced technology of schemes and sheaves. Furthermore, we work exclusively over the complex field for simplicity. However, most of the exposition we will give is identical over arbitrary fields.

Back onto the convergence of the two mathematical branches of symplectic geometry and algebraic geometry, it was discovered by Mumford, Guillemin, and Sternberg, that the symplectic reduction (in symplectic geometry) and the GIT quotient (in algebraic geometry) are intimately connected. This discovery was announced in Guillemin and Sternberg's 1982 paper titled *Geometric quantization and multiplicities of group representations* [GS82]. The connection that was found is that, under certain circumstances, the symplectic reduction and the GIT quotient are homeomorphic. We showcase this connection in the final chapter of this thesis (Chapter 5), where in particular, we prove the following theorem.

Theorem. (Guillemin and Sternberg). *Suppose that $X \subseteq \mathbb{CP}^n$ is a smooth projective G-variety, where G is the complexification of a compact connected real Lie group K. Denote the Fubini-Study moment map corresponding to the Hamiltonian K-manifold X by $\mu : X \to \mathfrak{K}^*$.*

With this setup, we have $\mu^{-1}(0) \subseteq G \cdot \mu^{-1}(0) = X^{ps}$, and this inclusion induces a homeomorphism

$$\mu^{-1}(0)/K \to X /\!\!/ G$$

between the symplectic quotient and the GIT quotient.

One of the main goals of this thesis is to prove the above theorem, which we refer to as the Guillemin-Sternberg Theorem.

We have organised the thesis into three distinct parts. Part I explores the world of symplectic geometry and its quotients. Specifically, it is first concerned with building the elementary theory of symplectic geometry in Chapter 1, before investigating the subtlety of quotients within symplectic geometry in Chapter 2. The next piece of the story, in Part II, explores the world of algebraic geometry and quotients in algebraic geometry, as given by GIT. We first review the fundamentals of classical algebraic geometry in Chapter 3 to allow for a discussion on Mumford's GIT in Chapter 4. Lastly, our story concludes in Part III with a unification of the ideas presented in Part I and Part II. It consists only of one chapter (Chapter 5), the crux of which is the Guillemin-Sternberg Theorem.

Throughout this thesis, we generally assume that the reader is familiar with some differential geometry, as we use its results throughout Part III. We also assume that the reader is familiar with tools and concepts from algebra up to say, an introductory course on modules and representation theory. Such knowledge is particularly desirable in Part II. Other than that, a working knowledge of linear algebra and basic point-set topology is assumed.

We have compiled a list of some non-standard results used throughout this thesis in the Appendix, for we felt that including them in the main chapters would distract too much from our discussion.

Here we state the notation we will use throughout this thesis. Suppose that X is a topological space and that G is a group acting on X. For a point $x \in X$ and a group element $g \in G$, we will frequently denote the action of g on x by $g \cdot x$. A *G-orbit* refers to the set of points obtained by acting the entire group G on a point x, which we denote by $G \cdot x$. While the *stabiliser of x*, denoted by G_x, refers to the subgroup of G given by the group elements which fix the point x (i.e., $g \in G$ such that $g \cdot x = x$). We remind the reader that the group action of G on X defines an equivalence relation on the points of X, where the equivalence classes are precisely the G-orbits. Thus, the set of G-orbits is a quotient space under this equivalence relation, equipped with the quotient topology. In this thesis, we refer to the set of G-orbits as the orbit space or the topological quotient, using these terms interchangeably. We denote the orbit space (or the topological quotient of X by G) as X/G, which should be read as 'X mod G'.

We denote smooth manifolds, symplectic manifolds, and algebraic varieties (affine or projective) with the letter X, to emphasise the link we draw between the symplectic and algebra-geometric worlds in Part III. We also denote arbitrary vector fields on smooth manifolds by the letter V, which is non-standard.

We denote a real Lie group that is *compact and connected* with the letter K, and we use the letter G for arbitrary Lie groups, algebraic groups, or for reductive groups. This is done for clarity, especially within the last chapter, where both compact Lie groups and reductive groups are present.

Let us begin!

Contents

Part I

Symplectic Geometry

Chapter 1

Symplectic Geometry

"But people thought about the subject from the point of view of classical mechanics, dating back to the nineteenth century. The main change in the field came with Gromov, when he came up with his amazing theorems. He started a revolution."

– Blaine Lawson, *Science Lives: Mikhail Gromov*, 2014. [14]

We begin our story with a tour of symplectic geometry, one of the core geometric theories in this thesis. Together with Riemannian geometry and Kähler geometry, symplectic geometry assembles a branch of differential geometry. Having arisen from the Hamiltonian formalism of classical mechanics, the field of symplectic geometry has close ties to physics. It remains an active area of research today, with applications ranging in partial differential equations and string theory.

The first chapter of this part of the thesis is devoted to exploring the fundamentals of symplectic geometry, first introducing differential forms on manifolds and their properties, before dipping into the relatively unknown (to Australian undergraduates, at least) subject of symplectic linear algebra. Once we learn the basics of symplectic linear algebra, we move on to looking at properties of symplectic manifolds, which are even-dimensional smooth manifolds equipped with a symplectic structure. The definitions and results in this chapter can largely be found in Lee [Lee12].

1.1 Differential Forms

We dedicate this section to establishing only what is necessary from the theory differential forms, that will be used for our discussion of symplectic geometry. We assume that the reader is familiar with the basics of differential geometry, but a good reference is Lee [Lee12]. Thus, all of the results in this section will be stated without proof. Although we will not investigate it here, differential forms happen to be a necessary ingredient for defining integration on manifolds; leading to a generalisation of Stoke's theorem from multivariable calculus.

We remind the reader that, for a smooth manifold X, the tangent bundle TX and cotangent bundle T^*X are given by the disjoint unions of the tangent spaces T_xX and cotangent spaces T_x^*X respectively, where the union is taken over all points of the manifold. Recall that a vector field on X is a map $x \mapsto v_x \in T_xX$, for $x \in X$, such that v_x varies smoothly as x varies over X. We denote the set of vector fields on X by $\mathfrak{X}(X)$.

Further recall that a $(0,k)$-tensor field (or covariant tensor field) on X is a mapping $x \mapsto \omega_x$, for $x \in X$, where $\omega_x : T_xX \times \ldots T_xX \to \mathbb{R}$ is multilinear over the real vector space T_xX, and such that ω_x varies smoothly as x varies over X. By convention, we set the $(0,0)$-tensor fields on X to be the smooth functions from X to $\mathbb{R}$, denoted by $C^\infty(X)$. The $(0,1)$-tensor fields on X are called 1-forms or covector fields, which we denote by $\Omega^1(X)$.

Definition 1.1.1. *A differential k-form is a $(0,k)$-tensor field ω on X that is totally skew-symmetric. That is, for each $x \in X$, and $v_1, \ldots, v_k \in T_xX$ we have*

$$\omega_x(v_1, \ldots, v_k) = \operatorname{sgn}(\sigma)\omega_x(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for every permutation σ of the symmetric group, S_k, on k letters. We call k the degree of ω.

In other words, permuting any of the variables of ω_x yields a sign change in the value of ω_x, depending on whether the permutation is even or odd.

For a differential k-form ω on X, we call k the degree of ω, and we denote the set of differential k-forms on X by $\Omega^k(X)$.

Theorem 1.1.2. *The set of differential k-forms, $\Omega^k(X)$, is a $C^\infty(X)$-module with addition and scalar multiplication defined pointwise. That is, given two differential k-forms ω and η, we define their sum by $(\omega+\eta)_x := \omega_x + \eta_x$, for $x \in X$, where*

$$(\omega_x + \eta_x)(v_1, \ldots, v_k) := \omega_x(v_1, \ldots, v_k) + \omega_x(v_1, \ldots, v_k),$$

for $v_1, \ldots, v_k \in T_xX$. Given a smooth function $f \in C^\infty(X)$, we define $f\omega$ by

$$(f\omega)_x := f(x)\omega_x,$$

for $x \in X$.

So, we can add differential forms of the same degree together, and scalar-multiply them by smooth functions. Recall that the tensor product of $\omega \in \Omega^k(X)$

and $\eta \in \Omega^l(X)$ is a $(0, k+l)$-tensor field defined by $x \mapsto (\omega \otimes \eta)_x$, for $x \in X$, where $(\omega \otimes \eta)_x$ is given by

$$(\omega \otimes \eta)_x(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) := \omega_x(v_1, \dots, v_k)\eta_x(v_{k+1}, \dots, v_{k+l}),$$

for $v_1, \dots, v_{k+l} \in T_xX$. Note that the tensor product of two differential forms is not a differential form in general. Nevertheless, the *wedge product* of two differential forms, which we now define, is itself a differential form.

Definition 1.1.3. *Let $\omega \in \Omega^k(X)$ and $\eta \in \Omega^l(X)$. The wedge product is a $C^\infty(X)$-bilinear map $\wedge : \Omega^k(X) \times \Omega^l(X) \to \Omega^{k+l}(X)$ defined by*

$$(\omega \wedge \eta)_x(v_1, \dots, v_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\omega \otimes \eta)_x(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})$$

for $x \in X$ and $v_1, \dots, v_{k+1} \in T_xX$. If $f, g \in \Omega^0(X)$ and $\omega \in \Omega^k(X)$, we set $f \wedge g := fg$ and $f \wedge \omega := f\omega =: \omega \wedge f$.

Recall the differential of a smooth function $f \in C^\infty(X)$ is a covector field given by

$$(df)_x(v) = vf$$

for $x \in X$ and $v \in T_xX$. We may write the differential on smooth functions as an $\mathbb{R}$-linear operator $d : \Omega^0(X) \to \Omega^1(X)$. In view of this, we extend the differential on smooth functions to an $\mathbb{R}$-linear operator on arbitrary differential k-forms, called the *exterior derivative.*

Theorem 1.1.4. *For each k, there is a unique $\mathbb{R}$-linear operator $d : \Omega^k(X) \to \Omega^{k+1}(X)$, called the exterior derivative, that satsifies the following properties:*

1. *If $\omega \in \Omega^k(X)$ and $\eta \in \Omega^l(X)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \eta \wedge \omega$;*
2. *The composition $d \circ d : \Omega^k(X) \to \Omega^{k+2}(X)$ is identically zero ($d^2 = 0$);*
3. *If $f \in \Omega^0(X) = C^\infty(X)$, then df is the differential of f.*

The exterior derivative generalises the differential operators of the curl and gradient from multivariable calculus to manifolds.

Given a smooth map $F : X \to Y$ between smooth manifolds X and Y, we can *pull-back* differential forms on Y to differential forms on X.

Definition 1.1.5. *For each k, the* pullback *by F is an $\mathbb{R}$-linear map $F^* : \Omega^k(Y) \to \Omega^k(X)$ given by*

$$(F^*\omega)_x(v_1, \dots, v_k) := \omega_{F(x)}(dF_x(v_1), \dots, dF_x(v_k))$$

for all $x \in X$ and $v_1, \ldots, v_k \in T_xX$.

Recall that $F : X \to Y$ is a smooth submersion if its derivative $dF_x : T_xX \to T_{F(x)}Y$ is surjective for every $x \in X$. Some important properties of the wedge product, the exterior derivative and the pullback, which we later use, include the following.

Proposition 1.1.6. *Let X, Y and Z be smooth manifolds, and $F : X \to Y$ and $G : Y \to Z$ be smooth maps. Then wedge product, exterior derivative and pullback satisfy the following properties:*

1. *If $F : X \to Y$ is a smooth submersion, then the pullback $F^* : \Omega^k(Y) \to \Omega^k(X)$ is injective;*
2. *If $\omega \in \Omega^k(Y)$ and $\eta \in \Omega^l(X)$, then $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$;*
3. *If $\omega \in \Omega^k(X)$ and $\eta \in \Omega^l(X)$, then $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$;*
4. *$(G \circ F)^* = F^* \circ G^*$;*
5. *If $\omega \in \Omega^k(Y)$, then $F^*(d\omega) = d(F^*\omega)$.*

There are two important classes of differential forms, which we now define.

Definition 1.1.7. *A differential k-form $\omega \in \Omega^k(X)$ is*

- exact *if $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(X)$;*
- closed *if $d\omega = 0$.*

By the property $d^2 = 0$ of the exterior derivative, exact forms are always closed. The Poincaré Lemma gives a partial converse:

Theorem 1.1.8 (Poincaré Lemma). *If ω is a closed form, then it is locally exact. That is, for each point $x \in X$ there is a neighbourhood $U \subseteq X$ containing x on which ω is exact.*

In general, the question of whether or not a closed form is globally exact depends on the global topology of the manifold at hand. A tool that is used to measure the failure of a closed form to be exact is called the *de Rham cohomology*.

Definition 1.1.9. *Denoting the space of closed k-forms by $Z^k(X)$ and the space of exact k-forms by $B^k(X)$, the kth de Rham cohomology group is the quotient vector space*

$$H^k(X) := Z^k(X)/B^k(X).$$

We introduce the de Rham cohomology here as it provides shorthand terminology for telling us when closed forms are globally exact, and we will use this shorthand later.[1] For instance, if we want to say that every closed 1-form on a smooth manifold X is exact, we will write $H^1(X) = 0$.

Note that the exterior derivative is an operator on differential forms that, in particular, increases their degree by one. In contrast, we define the interior product, which decreases the degree of differential forms by one.

Definition 1.1.10. *Let V be a vector field on X. The interior product is an $\mathbb{R}$-linear map $i_V : \Omega^k(X) \to \Omega^{k-1}(X)$ defined by*

$$(i_V\alpha)_x(v_1, \ldots, v_{k-1}) := \alpha(V_x, v_1, \ldots, v_{k-1})$$

for $x \in X$, $v_1, \ldots, v_{k-1} \in T_xX$, and $\alpha \in \Omega^k(X)$.

The interior product is an essential tool for us that will be heavily used in symplectic geometry. As a useful mnemonic, the interior product $i_V\alpha$ should be read as 'V into α'.

Another important tool to be used later is the *Lie derivative* of differential forms. Supposing X is a smooth manifold, we recall that every vector field on X generates a flow. Moreover, given two vector fields V and W on X, the *Lie derivative* of W with respect to V, is given by

$$(\mathcal{L}_V W)_x = \frac{d}{dt}\bigg|_{t=0} d(\Phi_{-t})_{\Phi_t(x)}(W_{\Phi_t(x)}) = \lim_{t\to 0} \frac{d(\Phi_{-t})_{\Phi_t(x)}(W_{\Phi_t(x)}) - W_x}{t}$$

where $x \in X$, and Φ is the flow generated by V.

We extend the Lie derivative to differential forms as follows.

Definition 1.1.11. *Let V be a vector field on X and let Φ be the flow generated by V. For a differential form ω, the Lie derivative of ω with respect to V is the differential form $\mathcal{L}_V\omega$ defined by*

$$(\mathcal{L}_V\omega)_x = \frac{d}{dt}\bigg|_{t=0} (\Phi_t^*\omega)_x = \lim_{t\to 0} \frac{d(\Phi_t)_x^*(\omega_{\Phi(t)}) - \omega_x}{t},$$

where $x \in X$.

As with the Lie derivative of vector fields, one should interpret the Lie derivative of a differential form ω with respect to a vector field V as a natural generalisation of

[1] A rather remarkable fact about the de Rham cohomology is that although it is defined using a smooth structure on a topological manifold, it only depends on the topology of the manifold! This makes it a powerful tool in differential geometry and differential topology, as one may study a manifold topologically by studying smooth structures on it.

the directional derivative, from several-variable calculus, of ω 'along the direction' of V.

With this interpretation in mind, we call ω *invariant* under the flow of V if $\mathcal{L}_V\omega = 0$. For flows that are defined globally over all of X, this is simply equivalent to the condition

$$\Phi_t^*\omega = \omega \text{ for all } t \in \mathbb{R},$$

where the diffeomorphisms $\Phi_t : X \to X$ are given in terms of the flow Φ generated by V.

So we have seen three operators on differential forms: the exterior derivative, the interior product, and the Lie derivative. Remarkably, they are not independent of each other. This is the content of 'Cartan's Magic Formula'.

Theorem 1.1.12 (Cartan's Magic Formula). *Let V be a vector field on X and let $\omega \in \Omega(X)$ be a differential form on X. The Lie derivative of ω wrt. V, the exterior derivative of ω, and the interior product of α wrt. V; are related by the following formula:*

$$\mathcal{L}_V\alpha = d(i_V\alpha) + i_V(d\alpha).$$

A straightforward consequence of this result, together with the fact $d^2 = 0$, is that the Lie derivative and exterior derivative operators commute on differential forms:

$$\mathcal{L}_V(d\omega) = d(i_V(d\omega)) = d(\mathcal{L}_V\omega).$$

Armed with the required knowledge of differential forms, we are almost ready to begin introducing symplectic structures on smooth manifolds. Before that however, we must first understand the basics of symplectic structures on vector spaces in the next section.

1.2 Symplectic Linear Algebra

In this section, we briefly tour symplectic structures on vector spaces, forming a subject called symplectic linear algebra. In particular, we examine a vector space analogue of an important concept we will develop in the final section of the next chapter. By the end of this section, we will be ready to tackle symplectic geometry. We begin with the definition of a symplectic form on a finite-dimensional real vector space V.

Definition 1.2.1. *A symplectic form* $\omega : V \times V \to \mathbb{R}$ *is a non-degenerate, skew-symmetric bilinear map.*[2] *We call the pair* (V, ω) *a symplectic vector space.*

We remind the reader that, for finite-dimensions, the non-degeneracy of ω means that the map $\widehat{\omega} : V \to V^*$ given by $v \mapsto \omega(\cdot, v)$ is a linear isomorphism; or equivalently, that $\omega(v, v') = 0$ for all $v' \in V$ implies that $v = 0$.

Non-degeneracy and skew-symmetry together imply that any matrix A representing the isomorphism $\widehat{\omega}$ is invertible and skew-symmetric. If V is odd-dimensional, then skew-symmetry implies that A has vanishing determinant which is impossible. Thus, symplectic vector spaces are necessarily *even-dimensional.*[3]

We now give a simple yet important example of a symplectic vector space that will come up again later.

Example 1.2.2 $(\mathbb{C}^n, \operatorname{Im} h)$**.** Consider the vector space $\mathbb{C}^n$, which is isomorphic to $\mathbb{R}^{2n}$ as a real vector space via the identification $z_k = q_k + ip_k$; and let $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denote the standard Hermitian inner product on $\mathbb{C}^n$ given by

$$h(z, w) = \sum_{k=1}^{n} z_k \overline{w}_k.$$

The imaginary part of h, denoted $\operatorname{Im} h$, defines a symplectic form ω on the vector space $\mathbb{C}^n$ (over $\mathbb{R}$). For instance, the skew-symmetry of $\omega = \operatorname{Im} h$ follows from

$$\omega(v, w) = \operatorname{Im} h(v, w) = \operatorname{Im} \overline{h(w, v)} = -\operatorname{Im} h(v, w) = -\omega(w, v)$$

for $v, w \in \mathbb{C}^n$; and as h is positive-definite and sesquilinear, for $0 \neq v \in \mathbb{C}^n$ we have

$$\omega(v, iv) = \operatorname{Im} h(v, iv) = \operatorname{Im}(-ih(v, v)) = -h(v, v) < 0,$$

so ω is non-degenerate. For $(q, p), (q', p') \in \mathbb{R}^{2n} \cong \mathbb{C}^n$ the symplectic form is given by

$$\omega((q, p), (q', p')) = \operatorname{Im} h(q + ip, q' + ip') = h(p, q') - h(q, p') = \sum_{k=1}^{n} (p_k q'_k - q_k p'_k).$$

Thus, the pair $(\mathbb{C}^n, \operatorname{Im} h)$ is a symplectic vector space.

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[2] One should note that the definition of a symplectic form mirrors that of the inner product. The differences are that the skew-symmetry and symmetry properties are interchanged, and that non-degeneracy is strengthened by positive-definiteness.

[3] This is in stark contrast to finite-dimensional inner product spaces, which can be any dimension!

If we consider a subspace W of a symplectic vector space (V, ω), it might not be symplectic unless it is even-dimensional. It will be vital not to have to restrict ourselves to even-dimensional subspaces, so we now define one important class of subspace in symplectic linear algebra that we require.[4]

Definition 1.2.3. *The subspace W is called isotropic if $\omega|_W \equiv 0$, or equivalently, $W \subseteq W^\omega$ where*

$$W^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$$

is a subspace of V called the symplectic orthogonal complement.[5]

The symplectic orthogonal complement is the symplectic analogue of the orthogonal complement in an inner product space, and it has similar properties such as those given in the following two propositions.

Proposition 1.2.4. *The subspace W satisfies* $\dim W + \dim W^\omega = \dim V$.

The proof of this simple result can be found in Lee [Lee12, Lemma 22.3]. The next proposition then follows quickly from this result but is given as an exercise in Lee, so we include its proof.

Proposition 1.2.5. *For a subspace W of a symplectic vector space (V, ω), the double complement $(W^\omega)^\omega = W$.*

Proof. The containment $W \subseteq (W^\omega)^\omega$ follows directly from the definition of W^ω. Applying the previous proposition to W and W^ω gives

$$\dim W + \dim W^\omega = \dim V = \dim W^\omega + \dim (W^\omega)^\omega,$$

and so $\dim W = \dim (W^\omega)^\omega$. Thus, $W = (W^\omega)^\omega$. $\square$

Now given an isotropic subspace W, we can form the quotient vector space W^ω / W. In fact, this quotient turns out to be a symplectic vector space.

Theorem 1.2.6. *Suppose W is an isotropic subspace of a symplectic vector space (V, ω). There exists a natural symplectic form on W^ω / W induced by ω.*

[4] There are other classes of subspaces in symplectic linear algebra, but we will not need them.

[5] We will be switching through different symplectic forms later, so this notation for the symplectic orthogonal complement that suggests dependence on ω is necessary.

Proof. The following proof is based off Lemma 23.3 in Ana [Sil01] with some more explanation included. Define a map $\omega_{W^\omega/W} : W^\omega/W \times W^\omega/W \to \mathbb{R}$ by

$$\omega_{W^\omega/W}(\pi(v), \pi(w)) = \omega(v, w)$$

for all $v, w \in W^\omega$, where $\pi : W^\omega \to W^\omega/W$ is the quotient morphism. We first show that this map is well-defined. Let $v, v' \in W^\omega$ and $w, w' \in W$. By the bilinearity of ω and the definition of W^ω we have

$$\omega(v + w, v' + w') = \omega(v, v') + \omega(v, w') + \omega(w, v') + \omega(w, w') = \omega(v, v')$$

where the last term vanishes since W is isotropic. Now we show that $\omega_{W^\omega/W}$ is symplectic. Bilinearity and skew-symmetry of $\omega_{W^\omega/W}$ follow directly from that of ω. To see non-degeneracy, suppose that $\omega_{W^\omega/W}(\pi(v), \pi(w)) = 0$ for all $\pi(v) \in W^\omega/W$ which implies that $\omega(v, w) = 0$ for all $v \in W^\omega$. Thus, $w \in (W^\omega)^\omega = W$ so $\pi(w) = 0_{W^\omega/W}$. □

The above theorem turns out to be a key ingredient in the proof of the Marsden-Weinstein-Meyer theorem at the end of Chapter 2, and so the quotient W^ω/W may be seen as a linear version of symplectic reduction. From now on, we will call such a quotient the *linear symplectic reduction.* Let us proceed on to looking at symplectic geometry.

1.3 Symplectic Manifolds

In this section, we explore the fundamental ideas in symplectic geometry; the study of symplectic structures on smooth manifolds. Although symplectic geometry has grown into a diverse field nowadays, here we only discuss the earliest constituents of the subject, before concluding with some remarks on symplectic geometry through the lens of classical mechanics. We begin with the definition of a symplectic manifold. Suppose X is a smooth manifold.

Definition 1.3.1. *A symplectic form is a closed differential* 2*-form* $\omega \in \Omega^2(X)$ *such that* ω_x *is non-degenerate on* T_xX *for all* $x \in X$. *We call the pair* (X, ω) *a symplectic manifold.*

In the above definition, ω_x is a symplectic form on the vector space T_xX for each $x \in X$ in the sense of Definition 1.2.1. In alignment with our discussion in the previous section, the definition of a symplectic manifold appears analogous to that of a Riemannian manifold. However, symplectic and Riemannian structures have rather different properties. For instance, symplectic manifolds are necessarily *even-dimensional*, due to their even-dimensional symplectic tangent spaces.

Perhaps the most important example of a symplectic manifold is the cotangent bundle of a smooth manifold.

Example 1.3.2. Consider the even-dimensional cotangent bundle T^*X, where X is a smooth manifold of dimension n. We construct a 1-form on T^*X, called the tautological 1-form, as follows. For a point on T^*X, denoted by (x, ξ), for $x \in X$ and $\xi \in T_x^*X$, we define the tautological 1-form $\tau \in \Omega^1(T^*X)$ by

$$\tau_{(x,\xi)} = d\pi^*_{(x,\xi)}\xi$$

where $\pi : T^*X \to X$ is the projection map and $d\pi^*_{(x,\xi)} : T_x^*X \to T^*_{\pi(x,\xi)}(T^*X)$ is the pointwise pullback of π.

Now we define a symplectic form ω on T^*X by $\omega = -d\tau$. Note that ω is certainly a closed 2-form as $d^2 = 0$. To see that $\omega_{(x,\xi)}$ is non-degenerate on each tangent space $T_{(x,\xi)}(T^*X)$, we proceed as follows. Consider the local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ induced by the local coordinates $(x_1, \ldots, x_n)$ in X. In these coordinates, $\xi \in T_x^*X$ has coordinate representation $\xi = \sum_{k=1}^n \xi_k dx_k$. Unravelling the definitions, we find that

$$\tau_{(x,\xi)} = \sum_{k=1}^n \xi_k \, dx_k$$

in these coordinates. Now, ω in these coordinates is given by

$$\omega_{(x,\xi)} = \sum_{k=1}^n dx_k \wedge d\xi_k.$$

This has the same form as the symplectic form in Example 1.2.2, so $\omega \in \Omega^2(T^*X)$ is a symplectic form, called the *canonical symplectic form* on T^*X.

In classical mechanics, the phase space of a physical system is frequently given by the cotangent bundle of some smooth manifold called the configuration space of the system.

The next example will be of utmost importance in Section 2.3, and throughout the rest of this thesis.

Example 1.3.3. Recall from Example 1.2.2, the symplectic vector space $(\mathbb{C}^n, \omega)$ where ω is the imaginary part of the standard Hermitian inner product h on $\mathbb{C}^n$.

Now $\mathbb{C}^n$ is a real smooth manifold of dimension $2n$ and so upon identification of $T_{(x,y)}\mathbb{C}^n$ with $\mathbb{C}^n$ itself, we may extend ω to a symplectic form on the smooth manifold $\mathbb{C}^n$. For coordinates (x, y) on $\mathbb{C}^n$, let $v = (q, p)$, $v' = (q', p') \in \mathbb{R}^{2n} \cong T_{(x,y)}\mathbb{C}^n$. Then, we have

$$\omega_{(x,y)}(v, v') = \sum_{k=1}^{n}(p_k q'_k - q_k p'_k) = \sum_{k=1}^{n}(dy_k(v)dx_k(v') - dx_k(v)dy_k(v')).$$

Thus, the symplectic form at $T_{(x,y)}\mathbb{C}^n$ is given by

$$\omega_{(x,y)} = \sum_{k=1}^{n} dy_k \wedge dx_k.$$

We can immediately see that ω is closed, and has the same form as the symplectic form in Example 1.2.2 so is degenerate. Therefore, the pair $(\mathbb{C}^n, \operatorname{Im} h)$ is a symplectic manifold.

Note that in coordinates, the symplectic form of the above example is equal to the canonical symplectic form on the cotangent bundle of a smooth manifold. This turns out to be no mere coincidence, as we will soon see in Darboux's Theorem.

Given two symplectic manifolds (X, ω) and (X', ω'), a diffeomorphism $F : X \to X'$ is called a *symplectomorphism* if

$$F^*\omega' = \omega.$$

A local diffeomorphism satisfying this condition is called a *local symplectomorphism.*

Theorem 1.3.4. (Darboux's Theorem) *Let (X, ω) be a symplectic manifold of dimension $2n$. There exists a local symplectomorphism from (X, ω) to $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. In particular, for any point $x \in X$, we can find local coordinates (called* Darboux coordinates*) $x_1, \ldots, x_n, y_1, \ldots, y_n$ around x such that*

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$$

with respect to these coordinates.

Darboux's Theorem highlights a drastic difference between symplectic geometry and Riemannian geometry. In the Riemannian geometry, a Riemannian metric can always be made to be the Euclidean metric at a particular point, but not necessarily around that point. This obstruction is due to curvature, which is a local invariant of Riemannian manifolds. In contrast, Darboux's Theorem says that symplectic manifolds have no local invariants. In other words, one cannot tell apart two different symplectic manifolds purely from studying them locally.

We decided to include Darboux's Theorem due to its fundamental importance in the field of symplectic geometry, but we omit the proof as we will not be using it frequently. The closure property of the symplectic form plays a crucial role in its proof, which may be found in Lee [Lee12, Theorem 22.13]

We now consider vector fields on symplectic manifolds. Suppose (X, ω) is a symplectic manifold. The non-degeneracy of the symplectic form ω implies that the linear maps $\widehat{\omega}_x : T_xX \to T_x^*X$ as defined at the beginning of Section 1.2, are vector space isomorphisms. Taking over the entire manifold, these linear maps together induce an isomorphism of vector fields on X and 1-forms on X:

$$\mathfrak{X}(X) \cong \Omega^1(X).$$

In other words, the vector fields on X are isomorphic to the 1-forms on X. So given a 1-form α on X, the non-degeneracy of ω gives a unique vector field V satisfying $i_V\omega = \alpha$.

Two particular classes of vector fields within symplectic geometry will be of importance to us.

Definition 1.3.5. *A vector field $V \in \mathfrak{X}(X)$ is*

- globally Hamiltonian *(or just* Hamiltonian*) if the 1-form $i_V\omega$ is exact;*
- symplectic *if the 1-form $i_V\omega$ is closed;*

If for any $x \in X$, there exists a neighbourhood U of x such that V is Hamiltonian, then V is called locally Hamiltonian*.*

We denote the set of Hamiltonian vector fields on (X, ω) by $\mathrm{Ham}(X, \omega)$. If a vector field $V \in \mathfrak{X}(X)$ is Hamiltonian, then by definition, there is a smooth function $H \in C^\infty(X)$ (called a *Hamiltonian associated to V*) such that

$$i_V\omega = dH. \tag{1.3.1}$$

Note that a Hamiltonian function may not be uniquely determined.

A symplectic vector field V is equivalently characterised by the condition $\mathcal{L}_V\omega = 0$, as

$$\mathcal{L}_V\omega = d(i_V\omega) + i_V(d\omega) = 0,$$

by Cartan's Magic Formula (Theorem 1.1.12), and the property $d\omega = 0$.

By the isomorphism of 1-forms and vector fields on a symplectic manifold (X, ω) described above, a smooth function $H \in C^\infty(X)$ induces a vector field V_H called the Hamiltonian vector field associated to H. Also, since exact forms are closed, any Hamiltonian vector field is a symplectic vector field. We have a partial converse as given in the following proposition.

Proposition 1.3.6. *A vector field V on X is symplectic if and only if V is locally Hamiltonian.*

Proof. Suppose V is locally Hamiltonian. Then for any point $x \in X$ there is a neighbourhood U of x and a smooth function $H \in C^\infty(U)$ such that $i_V\omega = dH$ so $i_V\omega$ is closed as $d^2 = 0$. Thus, V is symplectic. Conversely, if V is symplectic then by definition the 1-form $i_V\omega$ is closed. By the Poincaré Lemma (Lemma 1.1.8), each point $x \in X$ has a neighbourhood U on which the closed form $i_V\omega$ is exact and so there exists a smooth function $f \in C^\infty(U)$ such that $i_V\omega = df$ which implies that there is a vector field $V_f \in \mathfrak{X}(U)$ such that $i_{V_f}\omega = df$ so we have $i_V\omega = i_{V_f}\omega$. The non-degeneracy of ω then implies that $V = V_f$ on U so V is locally Hamiltonian. □

Proposition 1.3.7. *Every locally Hamiltonian vector field V on X is globally Hamiltonian if and only if $H^1(X) = 0$.*

Proof. Suppose that $H^1(X) = 0$, which is to say that every closed differential 1-form is exact, and that $V \in \mathfrak{X}(X)$ is locally Hamiltonian. Then by the previous proposition, V is symplectic, so the 1-form $i_V\omega$ is closed and so is also exact since $H^1(X) = 0$. Thus, V is globally Hamiltonian.

Conversely, suppose every locally Hamiltonian vector field is globally Hamiltonian and consider a closed differential 1-form $\alpha \in \Omega^1(X)$. By the non-degeneracy of ω, there exists a vector field $V \in \mathfrak{X}(X)$ such that $i_V\omega = \alpha$. We have $0 = d\alpha = d(i_V\omega)$ which implies that V is symplectic and therefore locally Hamiltonian by the previous proposition. Now by hypothesis, V is globally Hamiltonian and so there exists a smooth function $f \in C^\infty(X)$ such that $df = i_V\omega = \alpha$. Thus, α is exact, which proves that $H^1(X) = 0$. □

As a result of the previous two propositions, we have the following corollary.

Corollary 1.3.8. *If $H^1(X) = 0$, then symplectic vector fields are Hamiltonian.*

Later on, this corollary will prove useful for symplectic manifolds that have $H^1(X) = 0$, as we only need to check whether a vector field is symplectic.

We conclude this section with some brief remarks on symplectic geometry in its relation to classical mechanics. First, we derive *Hamilton's equations* for a *Hamiltonian flow.* Suppose (X, ω) is a symplectic manifold of dimension $2n$.

Definition 1.3.9. *The flow generated by a Hamiltonian vector field V on X is called the* Hamiltonian flow *associated with V.*

In Darboux coordinates, which we label by (q_k, p_k), a locally Hamiltonian vector field $V_H \in \mathfrak{X}(U)$ (where $(U, (q_k, p_k))$ is the Darboux chart) takes a rather simple form given by

$$V_H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} \right)$$

where $H \in C^\infty(X)$ is a Hamiltonian associated to V_H. To see this, let us verify equation (1.3.1) as follows:

$$\begin{aligned} i_{V_H}\omega &= \sum_{k=1}^{n} (dq_k \wedge dp_k)(V_H, \cdot) = \sum_{k=1}^{n} (dq_k(V_H)dp_k(\cdot) - dp_k(V_H)dq_k(\cdot)) \\ &= \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial q_k} dq_k \right) = dH. \end{aligned}$$

An integral curve $\gamma : I \subseteq \mathbb{R} \to U$ of the Hamiltonian vector field V_H, given by $\gamma(t) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ with $\gamma(0) = (q_k, p_k)$ must satisfy

$$\frac{dq_k}{dt}(t) = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt}(t) = -\frac{\partial H}{\partial q_k}$$

for all $k = 1, \ldots, n$. This is a system of ordinary differential equations called Hamilton's equations. If $X = T^*Q$ is the cotangent bundle of a smooth manifold Q, then it turns out that Hamilton's equations are equivalent to Newton's second law.

As stated earlier, many physical systems in classical mechanics have an associated configuration space Q which is a smooth manifold modelling the degrees of freedom of the system. Instead of studying the configuration space Q, Hamiltonian thought to consider the phase space, which is modelled by the cotangent bundle of Q, and puts position and momentum on equal footing.

A simple example is the physical system consisting of n unconstrained non-interacting particles in 3 dimensions has configuration space $\mathbb{R}^{3n}$, with each point representing a position of the n particles. The phase space is then $\mathbb{R}^{6n}$, with each point now representing a position and momentum of the n particles. For physical systems exhibiting constrained motion, the resulting phase space is typically a submanifold of ambient Euclidean space.

The role of the symplectic form on the phase space of a physical system is to encode the dynamics of the system in a way that is in accordance with Newton's laws of motion. The force exerted on a body is usually given by a potential function called the Hamiltonian, which one may think of as representing the energy of the system. The trajectory of a particle is given by the integral curves of a vector field in phase space that corresponds to the Hamiltonian. The non-degeneracy property of the symplectic form captures such a correspondence. Furthermore, the closure property of the symplectic form corresponds to the time-invariance of the laws of physics. In particular, the symplectic form should not vary over time along the integral curves of a Hamiltonian vector field, a requirement that is dependent on the closure property.

We have studied some of the early foundations of symplectic geometry. Of course, there is much more to this still actively researched subject that we could never hope to cover in this thesis. For instance, as hinted at in the epigraph of this chapter, Mikhael Gromov greatly transformed symplectic geometry. Nevertheless, having toured the essentials, we move onto the next chapter in which we look at group actions on symplectic manifolds.

Chapter 2

Quotients in Symplectic Geometry: Symplectic Reduction

"It would not be an exaggeration to say that the area of symplectic geometry which deals with global properties of Hamiltonian group actions is, to a large extent, a branch of equivariant topology."

– Guillemin et al., *Moment maps, cobordisms, and Hamiltonian group actions*, 2002 [Gui+02, p. 3].

In this chapter, we look at equivariant symplectic geometry; the study of group actions on symplectic manifolds and the quotient objects that arise from them. The theory of quotients in symplectic geometry is surprisingly subtle and cannot be arrived at quickly without first understanding quotients of *smooth* manifolds by group actions, which is itself a somewhat abstruse part of differential geometry. Hence, we dedicate Section 2.2 to give a detailed account of group actions on smooth manifolds, before delivering the theory of quotients in symplectic geometry, known as *symplectic reduction*, in Sections 2.3 amd 2.4. The group actions we consider in this chapter are given by Lie groups, so we first review some Lie theory in Section 2.1. The climax of the chapter is an application of symplectic reduction to complex projective spaces, the results of which will extend to Chapter 5.

2.1 Lie Theory

In this section we review prerequisite material from Lie theory, and so what is presented here is minimally expository. Instead of giving a full exposition with proofs and many examples, here we collate some definitions and results that will be used immediately in the proceeding sections of this chapter. The reader can find a more detailed exposition of Lie theory in Lee [Lee12] and Helgason [Hel78].

Definition 2.1.1. *A real/complex*[1] *Lie group G is a smooth/complex manifold with a compatible group structure. That is, the multiplication map and inversion maps are smooth/holomorphic maps.*

One may think of a Lie group as a group of symmetries that continuously vary, such as the symmetries of a sphere, in contrast to the discrete symmetries of a regular polygon whose group of symmetries are the finite dihedral groups.

Some important examples of Lie groups for us include:

- General linear group $\mathrm{GL}_n(\mathbb{R})$ ($\mathrm{GL}_n(\mathbb{C})$) of $n \times n$ invertible matrices with real (complex) entries;
- Multiplicative group of complex units $\mathbb{C}^* \cong \mathrm{GL}_1(\mathbb{C})$;
- Special linear group $\mathrm{SL}_n(\mathbb{C})$ of $n \times n$ invertible complex matrices with determinant one;
- Unitary group $\mathrm{U}(n)$ of $n \times n$ matrices satisfying $U^*U = I$;
- Circle group $U(1) \cong S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$;
- Special orthogonal group $\mathrm{SO}(n)$ of $n \times n$ matrices with real entries satisfying $Q^TQ = I$ and $\det Q = 1$.

We will primarily refer to these examples throughout this thesis. The appropriate definitions of subgroups and morphisms of Lie groups are given in the following two definitions.

Recall that a smooth map is an immersion if its derivative at every point is injective.

Definition 2.1.2. *A Lie subgroup H of a Lie group G is a subgroup of G with a smooth manifold structure such that the inclusion $\iota : H \hookrightarrow G$ is a smooth immersion.*

As an example, the unitary group $\mathrm{U}(n)$ is a Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Definition 2.1.3. *Let G and H be Lie groups. A Lie group homomorphism is a group homomorphism $\Phi : G \to H$ that is also a smooth map, and Φ is a Lie group isomorphism if it is a diffeomorphism.*

If G and H are isomorphic as Lie groups, we write $G \cong H$. Lie group homomorphisms with codomain $\mathrm{GL}(V)$, for a finite-dimensional real (complex) vector space V, are called real (complex) Lie group representations. An important example is the adjoint representation, which we define as follows.

Suppose G is a Lie group and consider the conjugation (by $g \in G$) map $C_g : G \to G$, which is a diffeomorphism. Taking its derivative at the identity

[1] This section will mostly deal with real Lie groups, but the definition of a complex Lie group is given for completion, and its use in Chapter 5. Unless stated otherwise, we will use the term 'Lie group' to refer to real Lie groups.

$e \in G$, we obtain an invertible linear map denoted by $\mathrm{Ad}_g : T_eG \to T_eG$.

Definition 2.1.4. *The* adjoint representation *of G on $\mathfrak{g}$ is the Lie group homomorphism* $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$ *given by* $g \mapsto \mathrm{Ad}_g$

Upon taking the derivative of the adjoint representation at the identity again, and identifying the tangent space $T_{\mathrm{Ad}(e)}(\mathrm{GL}(T_eG))$ with the vector space $\mathrm{End}(T_eG)$, we obtain the linear map $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$.

The group structure on a Lie group G allows us to study it infinitesimally by primarily looking at its tangent space at the identity. Thus, the tangent space at the identity is given a special name in Lie theory, called the *Lie algebra* of G.

Definition 2.1.5. *The* Lie algebra $\mathfrak{g}$ *of G, is the vector space T_eG together with the bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the* Lie bracket*, given by*

$$[A, B] := \mathrm{ad}(A)B$$

for $A, B \in \mathfrak{g}$.

Two important properties of the Lie bracket for the Lie algebra $\mathfrak{g}$ of G, are the following:

- Skew-symmetry: $[A, B] = -[B, A]$ for all $A, B \in \mathfrak{g}$;
- *Jacobi identity*: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$ for all $A, B, C \in \mathfrak{g}$.

We also define a Lie algebra abstractly as follows.

Definition 2.1.6. *A Lie algebra is a vector space V together with a skew-symmetric bilinear map $[\cdot,\cdot] : V \times V \to V$, called the Lie bracket, that satisfies the Jacobi identity.*

Working in this abstract setting for the moment, we define Lie subalgebras and Lie algebra homomorphisms.

Definition 2.1.7. *A Lie subalgebra $\mathfrak{h}$, of a Lie algebra $\mathfrak{g}$, is a subspace of $\mathfrak{g}$ such that $[A, B] \in \mathfrak{h}$ for all $A, B \in \mathfrak{h}$.*

For example, the Lie algebra of the Lie group $\mathrm{GL}_n(\mathbb{C})$ is the vector space $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries, with Lie bracket given by the matrix commutator $[A, B] := AB - BA$. The Lie algebra of the Lie group $\mathrm{U}(n)$, given by the skew-Hermitian (skew-adjoint) matrices

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A^* = -A\},$$

is a Lie subalgebra of the Lie algebra $M_n(\mathbb{C})$.

Definition 2.1.8. *Suppose that* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are Lie algebras. A* Lie algebra homomorphism *is a linear map* $\phi : \mathfrak{g} \to \mathfrak{h}$ *such that*

$$\phi([A, B]) = [\phi(A), \phi(B)]$$

for all $A, B \in \mathfrak{g}$.

For a Lie group G, the derivative of the adjoint representation of G, denoted by $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, is an example of a Lie algebra homomorphism. More generally, the derivative at the identity of a Lie group homomorphism defines a Lie algebra homomorphism.

One can show that elements of the Lie algebra are in one-to-one correspondence with vector fields that are *left-invariant*. That is, vector fields $A \in \mathfrak{X}(G)$ such that

$$d(L_g)_h(A_h) = A_{gh}$$

for all $g, h \in G$, where $L_g : G \to G$ is left multiplication. Left-invariant vector fields form a Lie algebra with Lie bracket given by the commutator of vector fields. Furthermore, every left-invariant vector field is complete and so generates a one-parameter subgroup or global flow on G. In other words, given a left-invariant vector field A on G, there exists a global flow denoted by

$$t \mapsto \exp(tA)$$

for $t \in \mathbb{R}$.

Definition 2.1.9. *Suppose G is a Lie group. The* exponential map $\exp : \mathfrak{g} \to G$ *is given by* $A \mapsto \exp(A)$*, where* exp *is the global flow on G generated by A.*

As an example, the Lie group $\mathrm{GL}_n(\mathbb{C})$ has exponential map given by

$$M_n(\mathbb{C}) \ni A \mapsto e^A := I + A + \frac{A^2}{2!} + \dots,$$

which is the matrix exponential.

From now on, we denote the Lie algebra of a Lie group G by either $\mathfrak{g}$ or $\mathrm{Lie}(G)$, and depending on context, will refer to T_eG or the left-invariant vector fields on G.

We now give two important results from Lie theory.

Theorem 2.1.10. *Suppose $\Phi : G \to H$ is a Lie group homomorphism and let $\phi = d\Phi_e : \mathfrak{g}\,(\cong T_eG) \to \mathfrak{h}\,(\cong T_eH)$ denote the corresponding Lie algebra homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow[\Phi]{} & H. \end{array}$$

An important example of this theorem is as follows. If $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional complex Lie group representation and $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ denotes its corresponding Lie algebra representation, then we have

$$e^{\rho_*(A)} = \rho(\exp(A)).$$

In particular, for the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ of a Lie group G on its Lie algebra $\mathfrak{g}$, we have $e^{\mathrm{ad}(A)} = \mathrm{Ad}(\exp(A))$.

Theorem 2.1.11. *Suppose that H is a Lie subgroup of a Lie group G. Then its Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{h}$ given by*

$$\mathfrak{h} = \{A \in \mathfrak{g} \mid \exp(tA) \in H \text{ for all } t \in \mathbb{R}\}$$

where $\exp : \mathfrak{g} \to G$ *is the exponential map for G.*

This theorem allows for easy computations of Lie algebras of Lie subgroups if we already know the Lie algebra of the larger Lie group.

We end this section with an important definition needed in Section 2.3. For a Lie group G, let $\mathfrak{g}^*$ denote the (vector space) dual of $\mathfrak{g}$.

Definition 2.1.12. *The coadjoint representation (or coadjoint action) of G on $\mathfrak{g}^*$ is the contragredient dual representation of the adjoint representation. That is, for every $g \in G$, $\eta \in \mathfrak{g}^*$ and $A \in \mathfrak{g}$ we have*

$$\mathrm{Ad}^*_g(\eta)(A) := \eta(\mathrm{Ad}_{g^{-1}}(A)).$$

Now that we have covered the prerequisite content from Lie theory, we are prepared to move onto the next section, where we look at actions of Lie groups on *smooth* manifolds and the theory of quotients of *smooth* manifolds by Lie groups.

2.2 Lie Group Actions on Smooth Manifolds: Quotient Manifolds

In this section, we give a thorough exposition of Lie group actions on smooth manifolds that culminates in the Quotient Manifold Theorem. In particular, we discuss the hypotheses needed in order for the topological quotient to have a smooth manifold structure. Here all proofs are given with full details, with the exception of the Quotient Manifold Theorem. Suppose G is a Lie group, and X is a smooth manifold.

Definition 2.2.1. *A smooth action of G on X is a (left) group action $\Phi : G \times X \to X$ that is a smooth map. For fixed $g \in G$, a smooth action Φ defines a diffeomorphism $\Phi_g : X \to X$ given by $x \mapsto \Phi(g, x)$. Furthermore, for fixed $x \in X$, we define the orbit map $\Phi^{(x)} : G \to X$ given by $g \mapsto \Phi(g, x)$.*

Let $\Phi : G \times X \to X$ be a smooth action of G on X. We denote $\Phi(g, x)$ by $g \cdot x$ if the action is not ambiguous in a given context.

From here on, we denote the G-orbit of a point $x \in X$ by $G \cdot x$ and the stabiliser of x by G_x.

We remind the reader of *G-equivariant* smooth maps. Suppose $F : X \to Y$ is a smooth map between smooth manifolds X and Y. If G acts smoothly on X and Y, then F is *G-equivariant* if

$$F(g \cdot x) = g \cdot F(x)$$

for every $x \in X$ and $g \in G$.

Knowing how a Lie group acts infinitesimally on a smooth manifold will prove useful later on in Section 2.3, so we here include the definition of the *infinitesimal action* associated to a smooth Lie group action on a smooth manifold.

Definition 2.2.2. *Let G be a Lie group acting smoothly on a smooth manifold X. For each $A \in \mathfrak{g}$, we define a vector field A_X on X by*

$$(A_X)_x = \frac{d}{dt}\Big|_{t=0} \exp(tA) \cdot x$$

where $\exp : \mathfrak{g} \to G$ *is the exponential map of G. The map $\mathfrak{g} \to \mathfrak{X}(X)$ given by $A \mapsto A_X$ is called the infinitesimal action of $\mathfrak{g}$ on X.*

As an application of Theorem 2.1.11, the Lie algebra of the stabiliser G_x is given by

$$\mathfrak{g}_x := \mathrm{Lie}(G_x) = \{A \in \mathfrak{g} \mid (A_X)_x = 0\}.$$

Given a smooth action of a Lie group G on a smooth manifold X, we remind the reader that the *orbit space* is the set of G-orbits endowed with the quotient topology, and is denoted by X/G. We would like to find suitable conditions for X/G to be a smooth manifold itself. It turns out the suitable conditions conditions are precisely when the action is smooth, free and proper. Recall that a map $f : X \to Y$ between topological spaces is proper if the preimage of every compact subset of Y is compact in X.

Definition 2.2.3. *The action of a Lie group G on a smooth manifold X is:*

- Free *if $G_x = \{e\}$ for every $x \in X$;*
- Proper *if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is proper.*

The goal of this section is to discuss and gain an understanding for why each of the hypotheses are needed in the Quotient Manifold Theorem.

Theorem 2.2.4 (**Quotient Manifold Theorem**)**.** *Let G be a Lie group acting properly, freely and smoothly on a smooth manifold X. The G-orbits are embedded submanifolds of X that are diffeomorphic to G. Moreover, the orbit space X/G can be given a unique smooth manifold structure with dimension $\dim X - \dim G$, such that the quotient map $\pi : X \to X/G$ is a smooth submersion.*

We will motivate each of the hypotheses in the Quotient Manifold Theorem by giving some non-examples. For this, we first recall some facts about the quotient topology and quotient maps.

Given a function $f : X \to Y$, a subset $U \subseteq X$ is said to be *saturated with respect to f* if $f^{-1}(f(U)) = U$. Note that we always have $U \subseteq f^{-1}(f(U))$. Now suppose that X is a topological space. Let $\sim$ be an equivalence relation defined on X, with corresponding quotient map $\pi : X \to X/{\sim}$, where $X/{\sim}$ denotes the set of equivalence classes in X.

We mention that the proofs of the following two propositions are completely our own.

Proposition 2.2.5. *Suppose that the equivalence class $[x]$ is dense in X for every $x \in X$, and that U is a saturated open set of X. Then U is either X or $\emptyset$.*

Proof. Suppose by way of contradiction that U is a saturated open set of X such that $U \neq \emptyset$ and $U \neq X$. Since $U \neq X$, we can take $y \in X \setminus U$. Since the equivalence class $[y]$ is dense in X, it intersects every non-empty open set of X. In particular we have $[y] \cap U \neq \emptyset$ as U is non-empty and open. Then there exists $x \in [y] \cap U$ so that $[x] = [y]$ and $x \in U$. This implies that $[x] = \pi(x) \in \pi(U)$ and so $[y] \in \pi(U)$. As U is saturated, $y \in \pi^{-1}(\pi(U)) = U$. But $y \in X \setminus U$ so we have a contradiction. $\square$

Proposition 2.2.6. *Let X be a topological space and let $\pi : X \to X/\sim$ be the quotient map. If the only saturated open sets in X are X and $\emptyset$ then $X/\sim$ has the trivial topology.*

Proof. Let V be open in $X/\sim$. Then $U = \pi^{-1}(V)$ is a saturated in X since from surjectivity of π we have $\pi(U) = \pi(\pi^{-1}(V)) = V$ and so $\pi^{-1}(\pi(U)) = \pi^{-1}(V) = U$. Moreover, U is open in X since π is continuous and V is open. By the hypothesis then we must have $U = X$ or $U = \emptyset$. This implies that $V = \pi(X) = X/\sim$ (since π is surjective) or $V = \pi(\emptyset) = \emptyset$. Thus, we are done. $\square$

Consider again a Lie group G acting smoothly on a smooth manifold X. The consequence of the previous two propositions is that if each G-orbit is dense, then the topology of the orbit space X/G is trivial and so cannot be Hausdorff.

Corollary 2.2.7. *Suppose that every orbit $G \cdot x$, for $x \in X$, is dense in X. Then the orbit space X/G has the trivial topology and so, in particular, cannot be Hausdorff.*

We give some examples to explain why all the hypotheses (smooth, free, proper) in the Quotient Manifold Theorem are needed. The following example shows that the assumption of 'free' is not necessary by itself.

Example 2.2.8. Let G be a group acting trivially on a smooth manifold X. Then for any $x \in X$, the stabiliser G_x is all of G and so the action is not free. However, the orbit space X/G is X, which is of course a smooth manifold by assumption.

The next example is one for which the action is free and whose quotient is a smooth manifold.

Example 2.2.9. Consider the action of the Lie group $\mathbb{R}^k$ on the smooth manifold $\mathbb{R}^k \times \mathbb{R}^n$ by translation in the $\mathbb{R}^k$-factor:

$$v \cdot (x, y) := (v + x, y).$$

Note that the action is free since $v \cdot (x, y) = (x, y)$ if and only if $v = 0$, and the orbits are affine subspaces parallel to $\mathbb{R}^k$. The orbit space $(\mathbb{R}^k \times \mathbb{R}^n)/\mathbb{R}^k$ then is homeomorphic to $\mathbb{R}^n$ which is a smooth manifold.

The next example is an exercise from Lee, whose details have been worked out by us, shows that the free hypothesis is not sufficient by itself.

Example 2.2.10 (Irrational winding on the torus). Let α be an irrational number and define an action of $\mathbb{R}$ on the 2-torus $\mathbb{T}^2 = S^1 \times S^1$ by

$$t \cdot (z, w) := (e^{2\pi i t} z, e^{2\pi i \alpha t} w).$$

To see that the action is free, suppose that $(z, w) \in \mathbb{T}^2$ is arbitrary and that $t \cdot (z, w) = (z, w)$. This implies that $e^{2\pi i t} = 1 = e^{2\pi i \alpha t}$ which in turns implies that $t = n$ and $\alpha t = m$ for integers $n, m \in \mathbb{Z}$. This is impossible unless $n = m = 0$ since α is irrational so $t = 0$ which shows that the action is free. We now show that the orbit space $\mathbb{T}^2/\mathbb{R}$ has the trivial topology which implies that it is not Hausdorff and so cannot be a manifold. We do this by showing that each orbit is dense in $\mathbb{T}^2$ and then apply Corollary 2.2.7. Consider the orbit of $(z, w) \in \mathbb{T}^2$ where $z = e^{2\pi i \theta}$ and $w = e^{2\pi i \varphi}$ and take $(z_1, z_2) = (e^{2\pi i s_1}, e^{2\pi i s_2}) \in \mathbb{T}^2$ to be arbitrary. We have

$$\begin{aligned}(s_1 - \theta + n) \cdot (z, w) &= (e^{2\pi i (s_1 - \theta)} e^{2\pi i \theta}, e^{2\pi i \alpha (s_1 - \theta)} e^{2\pi i \alpha n} e^{2\pi i \varphi}) \\ &= (z_1, e^{2\pi i (\alpha (s_1 - \theta) + \varphi))} e^{2\pi i \alpha n}).\end{aligned}$$

Now it suffices to prove that the set

$$S = \{e^{2\pi i \alpha n} \mid n \in \mathbb{Z}\}$$

is dense in S^1, since we can then approximate $\exp[2\pi i (s_2 - \alpha(s_1 - \theta) - \varphi)]$ by some $e^{2\pi i \alpha n}$ for large enough n. This would then imply that $(s_1 - \theta + n) \cdot (z, w)$ approximates (z_1, z_2), so that the orbit of (z, w) is dense in $\mathbb{T}^2$. To show that S is dense in S^1, take $\zeta = e^{2\pi i \theta} \in S^1$ and fix $\varepsilon > 0$. By Kronecker's approximation theorem there exists integers n, m such that $|n\alpha - m - \theta| < \varepsilon$ which gives

$$\left|e^{2\pi i \alpha n} - e^{2\pi i \theta}\right| = \left|e^{2\pi i (\alpha n - \theta)} - e^{2\pi i m}\right| \leqslant |2\pi(\alpha n - m - \theta)| < 2\pi\varepsilon.$$

Thus, S^1 is dense and we are done.

To avoid examples such as the above one, we require that the action is proper. We will show soon that the assumption of 'proper' implies that the orbit space is Hausdorff. In order to show this, we first prove some following topological facts.

Proposition 2.2.11. *If $f : X \to Y$ be a proper continuous map and Y is a locally compact Hausdorff topological space, then f is closed.*

Proof. Let $C \subseteq X$ be closed and let $y \in Y$ be a limit point of $f(C)$. To show that $f(C)$ is closed, we show that it contains y. Since Y is locally compact, there exists an open set U and a compact set K such that $y \in U \subseteq K$. Now y is also a limit point of $f(C) \cap K$ so it suffices to show that $f(C) \cap K$ is closed. Note that $f(C) \cap K = f(f^{-1}(K) \cap C)$ and since f is proper, $f^{-1}(K)$ is compact. As Y is

Hausdorff and K is compact, K is also closed so $f^{-1}(K)$ is closed by continuity of f. Since the intersection of closed sets is closed, we see that $f^{-1}(K) \cap C$ is a closed subset of the compact set $f^{-1}(K)$ and so $f^{-1}(K) \cap C$ is also compact. Again by continuity of f, $f(C) \cap K = f(f^{-1}(K) \cap C)$ is a compact subset of the Hausdorff space Y so is closed. Thus, $y \in f(C) \cap K \subseteq f(C)$ so we conclude that $f(C)$ is closed. □

The intuitive meaning of proper action may seem a little mysterious. Very roughly, a proper action says that 'most' of the group moves compact subsets entirely away from themselves.

Although the quotient map is not open in general, it is open for continuous topological group actions on topological spaces. We remind the reader that a topological group is a group with a topological structure such that the multiplication and inversion maps are continuous.

Proposition 2.2.12. *Let G be a topological group acting continuously on a topological space X with continuous action given by $\Phi : G \times X \to X$. Then the quotient map $\pi : X \to X/G$ is open.*

Proof. Let $U \subseteq X$ be open. We have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

where $g \cdot U = \{g \cdot x \mid x \in U\}$ for $g \in G$. Since the maps $\Phi_g : X \to X$ are homeomorphisms for every $g \in G$, we have that $g \cdot U = \Phi_g(U)$ is open for every $g \in G$. Thus, $\pi^{-1}(\pi(U))$ is open which implies that $\pi(U)$ is open since π is a quotient map. Therefore, π is an open map. □

The next proposition gives a sufficient condition for the orbit space to be Hausdorff.

Proposition 2.2.13. *Let G be a topological group acting continuously on a topological space X. If the set $A := \{(g \cdot x, x) \mid g \in G, x \in X\} \subseteq X \times X$ is closed then the orbit space X/G is Hausdorff.*

Proof. Suppose that the set A defined in the proposition is closed. Let $y_1 = G \cdot x_1$ and $y_2 = G \cdot x_2$ be distinct orbits in X/G, so that x_1 and x_2 are distinct points in X. This implies that $(x_1, x_2) \notin A$ since there is no $g \in G$ such that $g \cdot x_2 = x_1$ by assumption. Since A is closed, the set $X \times X \setminus A$ is open so there exists open neighbourhoods U and V of x_1 and x_2 respectively such that $(U \times V) \cap A = \emptyset$. Now $\pi(U)$ and $\pi(V)$ are open neighbourhoods of y_1 and y_2 respectively since π is open by the previous proposition. If $\pi(U) \cap \pi(V) \neq \emptyset$ then there exists $y \in \pi(U) \cap \pi(V)$ so there are $x \in U$ and $x' \in V$ such that $G \cdot x = G \cdot x'$ and so $(x, x') \in (U \times V) \cap A$ which is impossible. Thus, $\pi(U)$ and $\pi(V)$ are disjoint open neighbourhoods containing disjoint points y_1 and y_2 so X/G is Hausdorff. $\square$

Proposition 2.2.14. *If G is a Lie group acting continuously and properly on a smooth manifold X, then the orbit space X/G is Hausdorff.*

Proof. Denote the continuous group action by $\Phi : G \times X \to X$ so that the proper map is given by $\widetilde{\Phi} : G \times X \to X \times X$, $(g, x) \mapsto (\Phi(g, x), x)$ which is certainly continuous. Note that $X \times X$ is a locally compact Hausdorff space since X is a manifold and so $\widetilde{\Phi}$ is a closed map. Thus, the image of $G \times X$ under $\widetilde{\Phi}$ given by

$$\widetilde{\Phi}(G \times X) = \{(g \cdot x, x) \mid g \in G, \, x \in X\}$$

is closed. By Proposition 2.2.13, we conclude that X/G is Hausdorff. $\square$

Later on, we will be only be considering compact Lie groups, and in this case, their continuous actions are always proper.

Proposition 2.2.15. *Consider a continuous action Φ of a compact Lie group G on a smooth manifold X. Then Φ is proper.*

Proof. Suppose $K \subseteq X \times X$ is compact. We need to show that $\widetilde{\Phi}^{-1}(K)$ is compact in $G \times X$. To do so, it suffices to show that $\widetilde{\Phi}^{-1}(\pi_1(K) \times \pi_2(K))$ is compact, where $\pi_1 : X \times X \to X$ and $\pi_2 : X \times X \to X$ denote (continuous) projections onto the first and second factors respectively. This is because $\widetilde{\Phi}^{-1}(K)$ is a closed subset of $\widetilde{\Phi}^{-1}(\pi_1(K) \times \pi_2(K))$ and closed subsets of compact sets are compact. Now $\widetilde{\Phi}^{-1}(\pi_1(K) \times \pi_2(K))$ is a closed subset of $G \times \pi_2(K)$ which is compact since G and $\pi_2(K)$ are compact. Thus, $\widetilde{\Phi}^{-1}(\pi_1(K) \times \pi_2(K))$ is compact so Φ is proper. $\square$

The 'properness' assumption also implies that G-orbits are closed and that each stabiliser subgroup is compact. Thus, it is necessary for the G-orbits to be closed and for each stabiliser subgroup to be compact in order for the action to be proper.

Proposition 2.2.16. *If G is a Lie group acting continuously and properly on a smooth manifold X, then each orbit $G \cdot x$ is closed, and each stabiliser subgroup G_x is compact.*

Proof. Let $\Phi : G \times X \to X$ denote the continuous action of G on X. For the first claim, given $x \in X$, we need to show that the orbit map $\Phi^{(x)} : G \to X$ is closed. It suffices to prove that $\Phi^{(x)}$ is proper and continuous by Proposition

2.2.11. Fix $x \in X$ and let $K \subseteq X$ be compact. The continuity of $\Phi^{(x)}$ follows from the continuity of Φ. To see that $\Phi^{(x)}$ is proper, note that $K \times \{x\}$ is compact in $X \times X$ and so

$$\widetilde{\Phi}^{-1}(K \times \{x\}) = \{(g, x) \in G \times X \mid g \cdot x \in K\}$$

is compact since $\widetilde{\Phi}$ is proper. Let $\pi_1 : G \times X \to G$ denote the projection onto the first factor, which is continuous. Then

$$\pi_1(\widetilde{\Phi}^{-1}(K \times \{x\})) = \{g \in G \mid g \cdot x \in K\} = (\Phi^{(x)})^{-1}(K)$$

is compact and so $\Phi^{(x)}$ is proper. Thus, we have that $\Phi^{(x)}$ is a closed map and so the orbit $\Phi^{(x)}(G) = G \cdot x$ is closed in X.

For the second claim, we see that the stabiliser of $x \in X$ is given by

$$(\Phi^{(x)})^{-1}(\{x\}) = \{g \in G \mid \Phi^{(x)}(g) = g \cdot x = x\} = G_x$$

which is compact since $\Phi^{(x)}$ is proper (as proved in the first part of the proposition) and $\{x\}$ is compact in X. $\square$

The following example is of key importance to us and illustrates some subtle issues.

Example 2.2.17. Define a group action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ ($n \geq 1$) by scalar multiplication. The orbits are punctured lines through the origin, and the set of orbits is the quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ called complex projective space $\mathbb{CP}^n$; it has the quotient topology induced from the Euclidean topology on $\mathbb{C}^{n+1}$. A point in $\mathbb{CP}^n$ is denoted by $[z_0 : \cdots : z_n]$ for a nonzero representative $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$; points denoted this way are called homogeneous coordinates.

The above action is free on $\mathbb{C}^{n+1} \setminus \{0\}$ as the stabiliser of every point is trivial. However, if we include the origin, its stabiliser, given by $\mathbb{R}^*$, is not even compact, let alone trivial.

We construct a smooth manifold structure on $\mathbb{CP}^n$ as follows. Cover $\mathbb{CP}^n$ with $(n+1)$ open sets

$$U_i = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$$

for $i = 0, \ldots, n$, and define chart maps $\varphi_i : U_i \to \mathbb{C}^n$ by

$$[z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$$

with inverses $\varphi_i^{-1} : \mathbb{C}^n \to U_i$ given by

$$(z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \mapsto [z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n].$$

Clearly, both φ_i its inverse φ_i^{-1} are continuous maps for every $i = 0, \dots, n$ and so each φ_i is a homeomorphism and so $\mathbb{CP}^n$ is a topological manifold. To show that $\mathbb{CP}^n$ is smooth, we have to show that the transition maps $\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth for every $i, j = 0, \dots, n$. By symmetry, it suffices to show that ψ_{ij} is smooth for $i < j$. So we have

$$\begin{aligned} \psi_{ij}(z) &= \varphi_j([z_0 : \dots, z_{i-1} : 1 : z_{i+1} : \cdots : z_j : \cdots : z_n]) \\ &= \left(\frac{z_0}{z_j}, \dots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j} \right) \end{aligned}$$

where $z = (z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_j, \dots, z_n) \in \mathbb{C}^n$ such that $z_j \neq 0$. Each variable in the output of ψ_{ij} is a rational function and so ψ_{ij} is smooth. Thus, $\mathbb{CP}^n$ is a smooth manifold of real dimension $2n$.

Furthermore, there is a homeomorphism of $\mathbb{CP}^n$ with the orbit space S^{2n+1}/S^1 given by the action of the circle group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on the $2n+1$-sphere in $\mathbb{C}^{n+1}$. Rather than writing down an explicit map, we only explain the intuition behind the homeomorphism here. The idea is that the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C}^*$ takes place in two stages. First, we quotient by the multiplicative group of positive real numbers $\mathbb{R}^+$ which gives S^{2n+1}, and next quotient by the circle group S^1. Thus, this homeomorphism may be thought of as a polar decomposition of the action of $\mathbb{C}^*$:

$$(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \cong \frac{(\mathbb{C}^{n+1} \setminus \{0\}))/\mathbb{R}^+}{S^1} \cong S^{2n+1}/S^1.$$

This characterisation immediately highlights the compactness of $\mathbb{CP}^n$ as S^{2n+1} is compact and taking quotients preserves compactness.

We will show later that this homeomorphism is, in fact, a diffeomorphism and use the characterisation $\mathbb{CP}^n \cong S^{2n+1}/S^1$ to prove that $\mathbb{CP}^n$ is a symplectic manifold. This concludes the example.

For ease, we restate the Quotient Manifold Theorem.

Theorem 2.2.18 (Quotient Manifold Theorem). *Let G be a Lie group acting properly, freely and smoothly on a smooth manifold X. The G-orbits are embedded submanifolds of X that are diffeomorphic to G. Moreover, the orbit space X/G can be given a unique smooth manifold structure with dimension $\dim X - \dim G$, such that the quotient map $\pi : X \to X/G$ is a smooth submersion.*

We now have some sense for why each of the hypotheses in the Quotient Manifold Theorem are needed. Let us now consider an example that illustrates the first claim of the Quotient Manifold Theorem. Namely, that the G-orbits are embedded submanifolds of X that are diffeomorphic to G.

Example 2.2.19. Consider the action of $G = S^1$ on $X = \mathbb{C}\backslash\{0\}$ by multiplication: $z \cdot w = zw$. We may quickly observe that the G-orbits are circles of varying radii, centered at the origin. Thus, it is clear here that they are embedded submanifolds of X and that they are diffeomorphic to G by re-scaling. Furthermore, the orbit space X/G is homeomorphic to $(0, \infty)$, which is indeed a smooth manifold of dimension 1.

A complete proof of the Quotient Manifold Theorem is given in Theorem 21.10 of Lee [Lee12].

We conclude this section with a characterisation of the tangent spaces of a quotient manifold.

Proposition 2.2.20. *Let G be a Lie group acting properly, freely and smoothly on a smooth manifold X so that the orbit space X/G is a smooth manifold by the Quotient Manifold Theorem (Theorem 2.2.4), and that the quotient map $\pi : X \to X/G$ is a smooth submersion. Then for every $x \in X$, we have the following isomorphism*

$$T_{\pi(x)}(X/G) \cong T_xX/T_x(G \cdot x).$$

Proof. First note that since $\pi : X \to X/G$ is a smooth submersion, the derivative $d\pi_x : T_xX \to T_{\pi(x)}(X/G)$ is surjective and so $\operatorname{Im} d\pi_x = T_{\pi(x)}(X/G)$. From the first isomorphism theorem for vector spaces, it follows that

$$T_{\pi(x)}(X/G) \cong T_xX/\ker d\pi_x.$$

It remains to show that $\ker d\pi_x = T_x(G \cdot x)$. To achieve this, note that $G \cdot x$ is an embedded submanifold of X for every $x \in X$ according to the Quotient Manifold Theorem (Theorem 2.2.4). We identify $T_x(G \cdot x)$ with $\operatorname{Im} di_x \subseteq T_xX$ where $i : G \cdot x \hookrightarrow X$ is the inclusion map. Noting that $G \cdot x = \pi^{-1}(\pi(x))$ is a regular level set of π since π is a smooth submersion, it follows that $\pi \circ i$ is constant on $G \cdot x$ and so $d\pi_x \circ di_x : T_x(G \cdot x) \to T_{\pi(x)}(X/G)$ is identically zero. Thus, $\operatorname{Im} di_x \subseteq \ker d\pi_x$. Now from the rank-nullity theorem we have

$$\dim \ker d\pi_x = \dim T_xX - \dim T_{\pi(x)}(X/G) = \dim T_x(G \cdot x) = \dim \operatorname{Im} di_x,$$

where the second last equality follows from the fact that $G \cdot x$ is a regular level set of π. Therefore, $T_x(G \cdot x) = \ker d\pi_x$ which completes the proof. $\square$

This characterisation of the tangent spaces of a quotient manifold will prove vital in the final section of this chapter. Having considered the subtlety involved in the theory of quotients of *smooth* manifolds, we are now ready to delve into Lie group actions on *symplectic* manifolds.

2.3 Lie Group Actions on Symplectic Manifolds: The Moment Map

In this section, we turn our attention to Lie group actions on *symplectic* manifolds, and introduce the moment map associated with so-called Hamiltonian actions. To simplify the exposition of the next two sections, we assume that K is a real, *compact*, and *connected* Lie group acting on a symplectic manifold (X, ω). We have attempted to arrive at the definition of the moment map in a natural manner, via the dual comoment map, rather than merely stating the definition of a Hamiltonian action without motivation. We begin with the definition of a symplectic action.

Definition 2.3.1. *The action $\Phi : K \times X \to X$ of K on (X, ω) is symplectic if it acts via symplectomorphisms. In other words, if we have*

$$\Phi_k^* \omega = \omega$$

for every $k \in K$, where $\Phi_k : X \to X$ is the diffeomorphism given by $x \mapsto \Phi(k, x)$.

Put more simply, symplectic actions preserve the symplectic form. The corresponding notion of symplectic action at the level of the Lie algebra of K is an *infinitesimally symplectic* action. Recall that the infinitesimal action of K on X is a map $\phi : \mathfrak{K} \to \mathfrak{X}(X)$ sending each $A \in \mathfrak{K}$ to a vector field A_X given by

$$(A_X)_x = \left.\frac{d}{dt}\right|_{t=0} \exp(tA) \cdot x$$

for $x \in X$.

Definition 2.3.2. *A symplectic action of K on (X, ω) is infinitesimally symplectic if, for every $A \in \mathfrak{K}$, the vector field $A_X \in \mathfrak{X}(X)$ is a symplectic vector field. That is, the Lie derivative of ω with respect to A_X vanishes: $\mathcal{L}_{A_X}\omega = 0$ for every $A \in \mathfrak{K}$.*

As a small digression, the algebra of smooth functions $C^\infty(X)$ can be endowed with the structure of a Lie algebra via the symplectic form. We define the *Poisson bracket* $\{\cdot, \cdot\}$ of two smooth functions $f, g \in C^\infty(X)$ by

$$\{f, g\} := \omega(V_f, V_g)$$

where V_f and V_g are the Hamiltonian vector fields corresponding to f and g respectively. One can show that $(C^\infty(X), \{\cdot,\cdot\})$ is a Lie algebra, which we do not so here. A proof may be found in Lee [Lee12, Proposition 22.19].

Infinitesimally symplectic actions provide a method of studying how a Lie group acts infinitesimally on a symplectic manifold. Suppose now, for the infinitesimal action, we have that A_X is a Hamiltonian vector field on X for some $A \in \mathfrak{K}$. We remind the reader that this means the 1-form $i_{A_X}\omega$ is exact. That is, there exists a smooth function (not necessarily unique), which we denote by μ^A, satisfying Hamilton's equation:

$$d\mu^A = i_{A_X}\omega.$$

If this is the case for all $A \in \mathfrak{K}$, then there exists a lift (not necessarily unique) $\widetilde{\phi} : \mathfrak{K} \to C^\infty(X),\ A \mapsto \mu^A$ of the infinitesimal action $\phi : \mathfrak{K} \to \mathrm{Ham}(X,\omega) \subseteq \mathfrak{X}(X)$.[2] In other words, the following diagram commutes:

$$\begin{tikzcd} & C^\infty(X) \arrow[d] \\ \mathfrak{K} \arrow[ur, dashed, "\widetilde{\phi}"] \arrow[r, "\phi"'] & \mathrm{Ham}(X,\omega). \end{tikzcd}$$

Recall that the set of smooth functions $C^\infty(X)$ is a Lie algebra with Lie bracket given by the Poisson bracket:

$$\{f, g\} := \omega(V_f, V_g),$$

where V_f and V_g are the Hamiltonian vector fields associated to the smooth functions f and g respectively.

Definition 2.3.3. *Suppose that the action of K on (X,ω) is infinitesimally symplectic. If there exists a lifting $\widetilde{\phi} : \mathfrak{K} \to C^\infty(X)$ of the infinitesimal action that is a Lie algebra homomorphism; in other words,*

$$\{\mu^A, \mu^B\} = \mu^{[A,B]} \tag{2.3.1}$$

for $A, B \in \mathfrak{K}$, then the action of K on (X,ω) is called Hamiltonian.

If the action of real, compact and connected Lie group K on a symplectic manifold (X,ω) is Hamiltonian, we call X a Hamiltonian K-manifold.

[2] A lifting of the infinitesimal action to $\mathbb{C}^\infty(X)$ given here is sometimes called a comoment map in the standard literature.

Note that the left side of Equation 2.3.1 can be rewritten as $\{\mu^A, \mu^B\} = (d\mu^A)(B_X) = B_X(\mu^A)$ as μ^A and μ^B are Hamiltonian functions for the Hamiltonian vector fields A_X and B_X respectively.

Given a Hamiltonian K-manifold, it will be convenient for us to work with the dual notion of a lifting $\widetilde{\phi} : \mathfrak{K} \to C^\infty(X)$, given as follows. We may define a smooth map $\mu : X \to \mathfrak{K}^*$ by $\mu(x)(A) = \widetilde{\phi}(A)(x) = \mu^A(x)$. It is shown in [Hec13, p.73] that, as K is connected, $\widetilde{\phi}$ being a Lie algebra homomorphism implies that μ is K-equivariant with respect to the action of K on X and the coadjoint action of K on $\mathfrak{K}^*$:

$$\mu(k \cdot x) = \operatorname{Ad}_k^* \mu(x)$$

for $x \in X$ and $k \in K$. Conversely, suppose that μ is K-equivariant. For $A, B \in \mathfrak{K}$ and $x \in X$, we have

$$\{\mu^A, \mu^B\}(x) = [(B_X)_x](\mu^A) = \left.\frac{d}{dt}\right|_{t=0} \mu^A(\exp(tB) \cdot x) = \left.\frac{d}{dt}\right|_{t=0} \mu(\exp(tB) \cdot x)(A).$$

Now by K-equivariance of μ, and Theorem 2.1.10, we obtain

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mu(\exp(tB) \cdot x)(A) &= \left.\frac{d}{dt}\right|_{t=0} (\operatorname{Ad}^*_{\exp(tB)} \mu(x))(A) = \left.\frac{d}{dt}\right|_{t=0} \mu(x)(\operatorname{Ad}_{\exp(-tB)} A) \\
&= \mu(x)\left(\left.\frac{d}{dt}\right|_{t=0} \exp(-t\,\operatorname{ad}B)A\right) = \mu(x)([A, B]) \\
&= \mu^{[A,B]}(x),
\end{aligned}$$

where the second-last line follows from the definition of the Lie bracket given in Definition 2.1.5. Thus, $\widetilde{\phi}$ is a Lie algebra homomorphism. With this in mind, we now define the moment map.

Definition 2.3.4. *Suppose that (X, ω) is a Hamiltonian K-manifold. A moment map is a K-equivariant smooth map $\mu : X \to \mathfrak{K}^*$ such that, for each $A \in \mathfrak{K}$, the smooth functions $\mu^A \in C^\infty(X)$ defined by $\mu^A(x) := \mu(x)(A)$ for $x \in X$ satisfy Hamilton's equation:*[3]

$$d\mu^A = i_{A_X}\omega. \tag{2.3.2}$$

As mentioned above, the K-equivariance assumption on the moment map is equivalent to the condition that the lift $\widetilde{\phi} : \mathfrak{K} \to C^\infty(X)$ is a Lie algebra homomorphism. So, the data of a moment map for a symplectic action of K on (X, ω) is equivalent to the action being Hamiltonian.

[3] We will also refer to this as the Hamiltonian condition.

One may think of the moment map as a collection, or vector, of Hamiltonians that are parameterised by the Lie algebra.

The moment map is named as such due to its origins in physics. For the translation group acting on $T^*\mathbb{R}^3$, the moment map is the linear momentum.[4] While for the rotation group $\mathrm{SO}(3)$ acting on $T^*\mathbb{R}^3$, the moment map is the angular momentum.

We end this section with a key example of a moment map, whose importance is emphasised as the following theorem.

Theorem 2.3.5. *Let K be a compact, connected Lie group acting on the symplectic manifold $(\mathbb{C}^n, \operatorname{Im} h)$, where $\operatorname{Im} h$ denotes the imaginary part of the standard inner product on $\mathbb{C}^n$, via a unitary representation $\rho : K \to U(n)$. This action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{K}^*$ given by*

$$\mu^A(z) = \frac{1}{2i} h(\rho_*(A)z, z)$$

for $A \in \mathfrak{K}$ and $z \in \mathbb{C}^n$, where $\rho_* : \mathfrak{K} \to \mathfrak{u}(n)$ denotes the Lie algebra representation corresponding to ρ.

Proof. The following proof is based on that of Heckman [Hec13, Theorem 5.11], with many of the omitted details included. First note that the symplectic form $\omega = \operatorname{Im} h$ is K-invariant as h is $U(n)$-invariant; thus the action is symplectic.

We first show that μ is K-equivariant. That is, for every $k \in K$ and $z \in \mathbb{C}^n$ we show that

$$\mu(k \cdot z) = \operatorname{Ad}_k^* \mu(z)$$

or equivalently,

$$\mu^A(\rho(k)z) = \mu^{\operatorname{Ad}_{k^{-1}} A}(z)$$

for every $A \in \mathfrak{K}$. Thus, it suffices to show that

$$h(\rho_*(A)\rho(k)z, \rho(k)z) = h(\rho_*(\operatorname{Ad}_{k^{-1}} A)z, z)$$

As ρ is a unitary representation and h is $U(n)$-invariant, we obtain

$$h(\rho_*(A)\rho(k)z, \rho(k)z) = h(\rho(k)^{-1}\rho_*(A)\rho(k)z, z) = h(\operatorname{Ad}_{\rho(k^{-1})}\rho_*(A)z, z)$$

[4]The translation group is not compact, but the notion of a moment map still exists for non-compact Lie groups.

where $\mathrm{Ad}_{\rho(k^{-1})} \in \mathfrak{gl}(\mathfrak{u}(n))$.[5] We denote conjugation by k^{-1} on K, and conjugation by $\rho(k^{-1})$ on $U(n)$, by $C_{k^{-1}}$ and $C_{\rho(k^{-1})}$ respectively. Then, as ρ is a representation, we obtain the equality

$$\rho \circ C_{k^{-1}} = C_{\rho(k^{-1})} \circ \rho,$$

and differentiating both sides of this expression gives

$$\mathrm{Ad}_{\rho(k^{-1})} \circ \rho_* = \rho_* \circ \mathrm{Ad}_{k^{-1}};$$

recalling that the adjoint representation is the derivative of the conjugation map on a Lie group. Thus, μ is K-equivariant.

Now we show that μ satisfies the Hamiltonian condition as follows. Let $z \in \mathbb{C}^n$ and note that $T_z\mathbb{C}^n \cong \mathbb{C}^n$. For any $A \in \mathfrak{K}$, we have

$$\begin{aligned}
(d\mu^A)_z(w) &= \frac{d}{dt}\bigg|_{t=0} \mu^A(z+tw) \\
&= \frac{1}{2i}\frac{d}{dt}\bigg|_{t=0} h(\rho_*(A)(z+tw), z+tw) \\
&= \frac{1}{2i}\frac{d}{dt}\bigg|_{t=0} [h(\rho_*(A)z, z) + th(\rho_*(A)z, w) \\
&\qquad\qquad + th(\rho_*(A)w, z) + t^2 h(\rho_*(A)w, w)] \\
&= \frac{1}{2i}[h(\rho_*(A)z, w) - \overline{h(\rho_*(A)z, w)}] = \operatorname{Im} h(\rho_*(A)z, w)
\end{aligned}$$

where in the last line we have used the fact that $\rho_*(A) \in \mathfrak{u}(n)$ is skew-Hermitian. Now observe by Theorem 2.1.10 that

$$(A_{\mathbb{C}^n})_z = \frac{d}{dt}\bigg|_{t=0} \exp(tA)\cdot z = \frac{d}{dt}\bigg|_{t=0} \rho(\exp(tA))z = \frac{d}{dt}\bigg|_{t=0} e^{t\rho_*(A)}z = \rho_*(A)z$$

and so we obtain $d\mu^A = i_{A_{\mathbb{C}^n}}\omega$. Therefore the action is Hamiltonian and μ is a moment map for the action. □

We remark that the moment map given above is not unique. Rather, it is unique up to addition by an element $\eta \in \mathfrak{K}^*$; that is fixed by the coadjoint action of K on $\mathfrak{K}^*$. To see why, denote the shifted moment map by $\widetilde{\mu} = \mu + \eta$. Then as μ is K-equivariant, we have

$$\mathrm{Ad}_k^*\widetilde{\mu}(z) = \mathrm{Ad}_k^*\mu(z) + \mathrm{Ad}_k^*\eta = \mu(k\cdot z) + \eta = \widetilde{\mu}(k\cdot z)$$

[5] Note the abuse of notation, as Ad means two different things in this proof.

so $\widetilde{\mu}$ is K-equivariant. The Hamiltonian condition for $\widetilde{\mu}$ trivially holds since the exterior derivative of a real constant vanishes.

Theorem 2.3.5 provides a stepping stone for important results in the next section, where we utilise the key notion of the moment map and dive into the subtle theory of quotients in symplectic geometry.

2.4 Symplectic Reduction

Here we give a suitable definition of a symplectic quotient, which was first considered by Marsden and Weinstein in [MW74], and by Meyer in [Mey73]. This section is a little different from others in the sense that we try to guide the reader through a derivation of why the symplectic quotient is defined the way it is. This is in contrast to most other texts that appear to take the approach of simply stating the seemingly mysterious definition and leave the main ideas somewhat obscure.

Given a real Lie group K acting smoothly, properly and freely on a symplectic manifold (X, ω), we näively consider the orbit space X/K and ask whether it is a symplectic manifold. We know it to be at least a smooth manifold according to the Quotient Manifold Theorem (Theorem 2.2.4), but is it symplectic? In other words, does the quotient manifold X/K *inherit*[6] a symplectic structure from X?

If it does, we may as well take the orbit space X/K as the definition of a symplectic quotient. However, a significant obstruction is the possibility that X/K may not be even-dimensional! Recall, by the quotient manifold theorem, the dimension of X/K is given by $\dim X - \dim K$ which need not be even.

So a more sophisticated definition is clearly needed. In parallel with the definition of the *linear symplectic reduction* from the end of Section 1.2, we consider subquotients of symplectic manifolds. That is, before quotienting by the group K, we first restrict to a submanifold of X. The key ingredient is the existence of a moment map, from the previous section. For now, we will work towards the following definition.

Definition 2.4.1. *Suppose (X, ω) is a Hamiltonian K-manifold. The symplectic reduction (or symplectic quotient) of X by K is the orbit space quotient*

$$X^{\text{red}} = \mu^{-1}(0)/K$$

where $\mu : X \to \mathfrak{K}^*$ is a moment map for the Hamiltonian action.

One might suspect that the tangent spaces of the symplectic quotient are symplectic vector spaces that are given by linear symplectic reductions. Indeed, this turns out to be true, as we will later show.

[6] We will explain what is meant by the term 'inherit' soon.

Suppose now that the action of a real Lie group K on a symplectic manifold (X, ω) is Hamiltonian, and let $\mu : X \to \mathfrak{K}^*$ denote a moment map for the action. One of the problems when defining a symplectic structure on the usual quotient space X/K is that X may be 'too large'.

Instead, we consider submanifolds of X given by the level sets of μ. That is, the level sets $\mu^{-1}(\eta)$ for arbitrary $\eta \in \mathfrak{K}^*$. Physically, these level sets corresponds to considering particle trajectories on which μ is a conserved quantity. In order to take the quotient of the level set $\mu^{-1}(\eta)$ by K, the level set must be K-invariant. In other words, we require that $\mu(k \cdot x) = \eta$ for every $k \in K$ and $x \in \mu^{-1}(\eta)$.

Proposition 2.4.2. *If $\eta \in \mathfrak{K}^*$, then $\mu^{-1}(\eta)$ is K-invariant if and only if η is fixed by the coadjoint action of K on $\mathfrak{K}^*$.*

Proof. Let $x \in \mu^{-1}(\eta)$ and $k \in K$. By the K-equivariance property in the definition of the moment map, we have

$$\mu(k \cdot x) = \mathrm{Ad}_k^* \mu(x) = \mathrm{Ad}_k^* \eta.$$

Hence, $k \cdot x \in \mu^{-1}(\eta)$ for all $k \in K$ if and only if η is a fixed point of the coadjoint action of K on $\mathfrak{K}^*$. □

Now, provided that $\eta \in \mathfrak{K}^*$ is fixed by the coadjoint action, the orbit space quotient $\mu^{-1}(\eta)/K$ is well-defined. This is precisely the candidate definition for a symplectic quotient first considered by Marsden and Weinstein [MW74], and Meyer [Mey73].

Definition 2.4.3. *Suppose (X, ω) is a Hamiltonian K-manifold and let $\mu : X \to \mathfrak{K}^*$ be a moment map. Let $\eta \in \mathfrak{K}^*$ be fixed by the coadjoint action of K on $\mathfrak{K}^*$. The* symplectic reduction *(or* symplectic quotient*) of X by K at η, is the orbit space quotient*

$$X_\eta^{red} = \mu^{-1}(\eta)/K.$$

As the point $0 \in \mathfrak{K}^*$ is always fixed by the coadjoint action, from now on we only consider the symplectic reduction at zero, and call this the symplectic reduction of X by K, denoted by X^{red}. We will soon show that the symplectic reduction is in fact a symplectic manifold under suitable conditions.

We now look at the conditions for which X^{red} is a smooth manifold, using results from Section 2.2 on quotient manifolds. First, $\mu^{-1}(0)$ needs to be an embedded submanifold of X and so it suffices for $0 \in \mathfrak{K}^*$ to be a regular value of μ, by the Regular Level Set Theorem (Appendix). We then also have that $\mu^{-1}(0)$ has dimension $\dim X - \dim K$ which may not be even (so of course, the zero level set may not be a symplectic manifold). The next proposition says that if K acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ is automatically a submanifold of X.

Proposition 2.4.4. *If K acts freely on $\mu^{-1}(0)$ then 0 is a regular value of μ.*

Proof. Let $x \in \mu^{-1}(0)$ and recall that

$$\operatorname{Ann}(\operatorname{Im} d\mu_x) := \{A \in \mathfrak{K} \mid [d\mu_x(v)](A) = 0 \text{ for all } v \in T_xX\}.$$

We claim that every $A \in \operatorname{Ann}(\operatorname{Im} d\mu_x)$ has $A_{X,x} = 0$. By the definition of the moment map, $A \in \operatorname{Ann}(\operatorname{Im} d\mu_x)$ implies that $\omega_x(A_{X,x}, v) = 0$ for every $v \in T_xX$. From this it follows that $A_{X,x} = 0$ by the non-degeneracy of ω_x. This completes our claim. Note that the curve $\gamma : \mathbb{R} \to X$ given by $\gamma(t) = x$ for all $t \in \mathbb{R}$ is an integral curve of A_X passing through x. As the curve $\gamma_A(t) = \exp(tA) \cdot x$ is also an integral curve of A_X passing through x, we have $\exp(tA) \cdot x = x$ and so $\exp(tA) \in K_x$ for every $t \in \mathbb{R}$. But K_x is trivial since the action is free on $\mu^{-1}(0)$ which proves that $A = 0$. Thus, $\operatorname{Ann}(\operatorname{Im} d\mu_x) = 0$ and so $\operatorname{Im} d\mu_x = \mathfrak{K}^*$, proving that $d\mu_x$ is surjective for every $x \in \mu^{-1}(0)$. Hence, $0 \in \mathfrak{K}^*$ is a regular value of μ.
□

Now, from the Quotient Manifold Theorem (Theorem 2.2.4), in order for X^{red} to be a smooth manifold, we require that the action is proper and free on $\mu^{-1}(0)$. In this case, X^{red} is a smooth manifold of (even) dimension $\dim X - 2\dim K$, as $\dim X$ is even. Since its dimension is even, can X^{red} now be given a symplectic structure inherited from X? It turns out the answer is yes, with no further assumptions! Now we explain our loose use of the term 'inherited' above.

Definition 2.4.5. *We say that a symplectic form ω on X^{red} is* inherited *from the symplectic form ω on X if $\iota^*\omega = \pi^*\omega^{red}$ where $\iota : \mu^{-1}(0) \hookrightarrow X$ is the inclusion map and $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/K$ is the quotient map.*

To get a symplectic structure on X^{red}, we need to define a symplectic form on each of the tangent spaces of X^{red}. We would like to use Theorem 1.2.6 from Section 1.2 to achieve this. By Proposition 2.2.20, we have that

$$T_{\pi(x)}X^{\text{red}} \cong T_x(\mu^{-1}(0))/T_x(K \cdot x)$$

for every $x \in \mu^{-1}(0)$. Now if we can show that $T_x(\mu^{-1}(0))$ is the symplectic orthogonal complement of $T_x(K \cdot x)$, then by definition this would imply that $T_x(K \cdot x)$ is an isotropic subspace of the symplectic vector space (T_xX, ω_x). We may then apply Theorem 1.2.6.

Lemma 2.4.6. *The subspace $T_x(\mu^{-1}(0))$ of the symplectic vector space (T_xX, ω_x) is the symplectic orthogonal complement of $T_x(K \cdot x)$ for all $x \in X$. That is,*

$$T_x(\mu^{-1}(0)) = T_x(K \cdot x)^{\omega_x}$$

for all $x \in \mu^{-1}(0)$.

Proof. We first show that $T_x(K \cdot x)^{\omega_x} \subseteq T_x(\mu^{-1}(0))$ for each $x \in X$. Take v in $T_x(K \cdot x)^{\omega_x}$ so that in particular, $\omega_x(A_{X,x}, v) = 0$ for all $A \in \mathfrak{K}$, since $A_{X,x}$ belongs to $T_x(K \cdot x)$. Recalling Equation (2.3.2):

$$d\mu^A = i_{A_X}\omega,$$

from the definition of the moment map, we then have $v \in \ker d\mu_x$ and so $T_x(K \cdot x)^{\omega_x}$ is contained in $\ker d\mu_x$. Now, $\ker d\mu_x = T_x(\mu^{-1}(0))$ because $\mu^{-1}(0)$ is a regular level set. Upon counting dimensions, using Proposition 1.2.4 implies that

$$\dim T_x(K \cdot x)^{\omega_x} = \dim T_xX - \dim T_x(K \cdot x).$$

Since $K \cdot x$ is diffeomorphic to K by the Quotient Manifold Theorem (Theorem 2.2.4), and $\mu^{-1}(0)$ has dimension $\dim X - \dim K$, we then have

$$\dim T_x(K \cdot x)^{\omega_x} = \dim X - \dim K = \dim T_x(\mu^{-1}(0))$$

and so we conclude that $T_x(\mu^{-1}(0)) = T_x(K \cdot x)^{\omega_x}$. $\square$

Now that $T_x(K \cdot x)$ is an isotropic subspace of T_xX for every $x \in \mu^{-1}(0)$, by Theorem 1.2.6 there exists a natural symplectic form $\omega^{\text{red}}_{\pi(x)}$ on $T_{\pi(x)}X^{\text{red}}$ such that

$$\omega^{\text{red}}_{\pi(x)}(d\pi_x(v), d\pi_x(w)) = \omega_x(v, w)$$

for all $v, w \in T_x(\mu^{-1}(0))$. Thus, we define a 2-form ω^{red} on X^{red} by $\pi(x) \mapsto \omega^{\text{red}}_{\pi(x)}$ and note that this is smooth since ω is smooth. By construction we have

$$(\pi^*\omega^{\text{red}})_x(v, w) = \omega^{\text{red}}_{\pi(x)}(d\pi_x(v), d\pi_x(w)) = \omega_x(v, w) = (\iota^*\omega)_x(v, w)$$

for every $x \in \mu^{-1}(0)$ and $v, w \in T_x(\mu^{-1}(0))$. Thus, $\iota^*\omega = \pi^*\omega^{\text{red}}$ and so ω^{red} is inherited from ω. It remains to show that ω^{red} is closed, or in other words, that the 3-form $d\omega^{\text{red}}$ vanishes. Since the pullback of a smooth map commutes with the exterior derivative, and ω is closed, we obtain

$$\pi^*(d\omega^{\text{red}}) = d(\pi^*\omega^{\text{red}}) = d(\iota^*\omega) = \iota^*(d\omega) = 0.$$

As $\pi : \mu^{-1}(0) \to X^{\text{red}}$ is a smooth submersion (by the Quotient Manifold Theorem), the pullback $\pi^* : \Omega^3(X^{\text{red}}) \to \Omega^3(\mu^{-1}(0))$ is injective and so we conclude that $d\omega^{\text{red}} = 0$.

We summarise the above discussion on symplectic quotients in the following theorem:

Theorem 2.4.7 (**Marsden-Weinstein-Meyer [MW74; Mey73]**). *Suppose that (X, ω) is a Hamiltonian K-manifold and let $\mu : X \to \mathfrak{K}^*$ be a moment map for this action. If K acts freely and properly on $\mu^{-1}(0)$ then the symplectic reduction $X^{red} = \mu^{-1}(0)/K$ is a symplectic manifold with a natural symplectic structure inherited from that of X.*

Within the realm of classical physics, the symplectic reduction of a phase space that describes some physical system is interpreted as the 'reduced' phase space. In other words, if a given physical system has a group of symmetries that leave it invariant (such as a rotations of a rigid body), it is often desirable to 'spend the symmetry' and perform calculations in the reduced phase space, which is often simpler to deal with. This method is in contrast to gauge-theoretic methods, in which one remains upstairs in the un-reduced phase space to perform calculations. For a more in-depth discussion on symplectic reduction in classical mechanics, we encourage the reader to seek out Butterfield's article *On Symplectic Reduction in Classical Mechanics* [But06].

We now look at an application of symplectic reduction to the construction of the complex projective space $\mathbb{CP}^n$, which is of utmost importance in later chapters. We remind the reader that a Lie group representation $\rho : K \to \mathrm{GL}_m(\mathbb{C})$ is *unitary* if its image lies in the unitary group $U(m) \subseteq \mathrm{GL}_m(\mathbb{C})$.

Example 2.4.8. Let $K = S^1 \cong U(1)$ act on the symplectic manifold $(\mathbb{C}^{n+1}, \operatorname{Im} h)$, where h denotes the standard inner product on $\mathbb{C}^{n+1}$, by

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n).$$

In other words, S^1 acts via a unitary representation $\rho : \lambda \mapsto \operatorname{diag}(\lambda, \ldots, \lambda)$. Then $\rho_*(A) = \operatorname{diag}(A, \ldots, A)$ for $A \in \operatorname{Lie} S^1$. By Theorem 2.3.5, there is a moment map for this action given by

$$[\mu(z)](A) = \frac{1}{2i} h(\mathrm{diag}(A, \dots, A)z, z) = \frac{1}{2i} h(Az, z) = \frac{A}{2i} h(z, z)$$

and under the identification $\mathfrak{u}(1)^* \cong \mathfrak{u}(1) \cong \mathbb{R}$ we have $\mu(z) = \frac{1}{2} h(z, z)$. As remarked after Theorem 2.3.5, we can shift the moment map by some $\eta \in \mathfrak{u}(1)^*$ that is fixed by the coadjoint action; but as $U(1)$ is abelian, every element on $\mathfrak{u}(1)^*$ is fixed by the coadjoint action. Thus, there is a family of moment maps for this action given by

$$\mu(z) = \frac{1}{2}(h(z, z) - C) = \frac{1}{2}\left(\sum_{k=0}^{n} |z_k|^2 - C\right)$$

where C is any real number. Upon taking $C = 1$, we observe that

$$\mu^{-1}(0) = \{z \in \mathbb{C}^{n+1} \mid \sum_{k=0}^{n} |z_k|^2 = 1\} = S^{2n+1}.$$

Now the action is free on $\mu^{-1}(0) = S^{2n+1}$ since at least one $z_i \neq 0$ on S^{2n+1} and so $\lambda z = z$ implies $\lambda = 1$; moreover, S^1 is compact and so the action is proper on $\mu^{-1}(0)$. Hence, by the Marsden-Weinstein-Meyer theorem (Theorem 2.4.7), the symplectic reduction of $\mathbb{C}^{n+1}$ by S^1 is a symplectic manifold of (real) dimension $\dim \mathbb{C}^{n+1} - 2 \dim S^1 = 2n$, given by

$$\mu^{-1}(0)/S^1 = S^{2n+1}/S^1 \cong \mathbb{CP}^n.$$

We call the induced symplectic form on $\mathbb{CP}^n$ the Fubini-Study form ω_{FS}; which satisfies $\pi^*\omega_{FS} = \iota^* \operatorname{Im} h$, where $\pi : S^{2n+1} \to S^{2n+1}/S^1$ is the quotient map and $\iota : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ is the inclusion map.

We conclude this chapter with an in-depth focus on Hamiltonian actions on $(\mathbb{CP}^n, \omega_{FS})$ that are induced by unitary representations on $\mathbb{C}^{n+1}$. Let K be a compact connected Lie group acting on the symplectic manifold $(\mathbb{C}^{n+1}, \omega)$, where $\omega = \operatorname{Im} h$ is the standard symplectic form on $\mathbb{C}^{n+1}$, via a unitary representation $\rho : K \to U(n+1)$. Denote an element of $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ by the equivalence class $[z]$ for some nonzero representative $z \in \mathbb{C}^{n+1}$.

Theorem 2.4.9. *The unitary representation $\rho : K \to U(n+1)$ on $\mathbb{C}^{n+1}$ induces a Hamiltonian action of K on the symplectic manifold $(\mathbb{CP}^n, \omega_{FS})$, where ω_{FS} is* *the Fubini-Study form, with moment map* (called the Fubini-Study moment map) $\mu' : \mathbb{CP}^n \to \mathfrak{K}^*$ *given by*

$$\mu'^A([z]) = \frac{h(\rho_*(A)z, z)}{2ih(z, z)}$$

for $A \in \mathfrak{K}^*$ *and* $z \in \mathbb{C}^{n+1} \setminus \{0\}$.

Proof. The following proof is a heavily adapted version of Heckman and Hochs' proof in [HH12, Theorem 1.30], with some omitted details now included. There, they prove the theorem in more generality using numerous lemmas along the way. It is worth noting that one of the main ideas used in Heckman and Hochs' proof is commuting actions. This is a nuance which, as we shall see, turns out to be trivial for us. Here we instead prove the theorem more directly while borrowing some of the ideas from Heckman and Hochs.

We begin by defining an action of K on $\mathbb{CP}^n = S^{2n+1}/S^1$ by

$$k \cdot (S^1 \cdot z) := S^1 \cdot (\rho(k)z)$$

for $k \in K$ and $z \in S^{2n+1}$, where $S^1 \cdot z \in \mathbb{CP}^n$ denotes the equivalence class of $z \in S^{2n+1}$. Note that this action is well-defined since the action of S^1 and $U(n+1)$ on S^{2n+1} commute. By abuse of notation, denote both the action maps of $k \in K$ on S^{2n+1} and on $\mathbb{CP}^n$ by k; and let $\iota : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ and $\pi : S^{2n+1} \to \mathbb{CP}^n$ denote the inclusion and quotient maps respectively.

To show that the action is symplectic, it suffices to show that $\pi^*(k^*\omega_{FS}) = \pi^*\omega_{FS}$ since the pullback $\pi^* : \Omega(\mathbb{CP}^n) \to \Omega(S^{2n+1})$ is injective. Recall that the Fubini-Study form satisfies $\pi^*\omega_{FS} = \iota^*\omega$; so by the properties of pullbacks and the definition of the action of K on $\mathbb{CP}^n$ we obtain

$$\pi^*(k^*\omega_{FS}) = k^*(\pi^*\omega_{FS}) = k^*\iota^*\omega = \iota^*\omega = \pi^*\omega_{FS}$$

where the second last equality follows from $U(n+1)$-invariance of ω.

We now establish that the action of K on $(\mathbb{CP}^n, \omega_{FS})$ is Hamiltonian. From Theorem 2.3.5, the action of K on $(\mathbb{C}^{n+1}, \omega)$ is Hamiltonian with moment map $\mu : \mathbb{C}^{n+1} \to \mathfrak{K}^*$ given by

$$\mu^A(z) = \frac{1}{2i} h(\rho_*(A)z, z)$$

for $A \in \mathfrak{K}^*$ and $z \in \mathbb{C}^{n+1}$. We define a map $\mu' : S^{2n+1}/S^1 \to \mathfrak{K}^*$ via the composition

$$S^{2n+1} \overset{\iota}{\hookrightarrow} \mathbb{C}^{n+1} \xrightarrow{\mu} \mathfrak{K}^*;$$

that is, $\mu'(S^1 \cdot z) := \mu(z)$ for $z \in S^{2n+1}$, and this map is well-defined as h is S^1-invariant. The K-equivariance of μ' follows from that of μ:

$$\mu'(k \cdot (S^1 \cdot z)) = \mu'(S^1 \cdot (\rho(k)z)) = \mu(\rho(k)z) = \mathrm{Ad}_k^* \mu(z) = \mathrm{Ad}_k^* \mu'(S^1 \cdot z)$$

for $k \in K$ and $z \in S^{2n+1}$. Lastly, we claim that

$$d\mu'^A = i_{A_{\mathbb{CP}^n}} \omega_{FS}$$

where $A_{\mathbb{CP}^n}$ is the infinitesimal action of $A \in \mathfrak{K}^*$ on $\mathbb{CP}^n$. Again by the injectivity of $\pi^* : \Omega(\mathbb{CP}^n) \to \Omega(S^{2n+1})$, it suffices to show that

$$\pi^*(d\mu'^A) = \pi^*(i_{A_{\mathbb{CP}^n}} \omega_{FS}).$$

By the properties of the exterior derivative and pullbacks, we obtain

$$\pi^*(d\mu'^A) = d(\pi^* \mu'^A) = d(\mu^A \circ \iota) = \iota^* d\mu^A = \iota^*(i_{A_{\mathbb{C}^{n+1}}} \omega).$$

Now it is not hard to compute that

$$\iota^*(i_{A_{\mathbb{C}^{n+1}}} \omega) = i_{A_{S^{2n+1}}}(\iota^* \omega) = i_{A_{S^{2n+1}}}(\pi^* \omega_{FS}) = \pi^*(i_{A_{\mathbb{CP}^n}} \omega_{FS}).$$

Thus, the action of K on $\mathbb{CP}^n$ is Hamiltonian with moment map $\mu' : \mathbb{CP}^n \to \mathfrak{K}^*$. For $A \in \mathfrak{K}^*$ and $z \in S^{2n+1}$, we may write the moment map as

$$\mu'^A(S^1 \cdot z) = \mu^A(z) = \frac{1}{2i} h(\rho_*(A)z, z) = \frac{h(\rho_* A(z), z)}{2ih(z,z)}$$

where $h(z,z) = 1$. This formula now does not depend on the length of z and so we may write the moment map as

$$\mu'^A([z]) = \frac{h(\rho_*(A)z, z)}{2ih(z,z)}$$

for an equivalence class $[z] \in (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ where $z \in \mathbb{C}^{n+1}$ is a nonzero representative. This completes the proof. $\square$

The reader should keep in mind Theorem 2.4.9 above for the final chapter, in which it forms a vital foundation for the connection between the quotients in symplectic geometry and algebraic geometry. Here we have emphasised its importance by the calling it a theorem as opposed to an example, which may otherwise be more appropriate.

We have now completed the end of Part I, in which we have explored the foundations of symplectic geometry in Chapter 1, before discussing quotients by compacted, connected Lie groups in symplectic geometry within this chapter. We now depart from the world of symplectic geometry, and move onto the second part of this thesis, where we turn our attention to the seemingly disparate world of algebraic geometry.

Part II

Algebraic Geometry

Chapter 3

Classical Algebraic Geometry

"Should you just be an algebraist or a geometer?" is like saying "Would you rather be deaf or blind?"
– Michael Atiyah, *Mathematics in the 20th century*, 2001 [Ati01, p. 659].

This chapter marks the beginning of Part II of this thesis, in which we enter the domain of algebraic geometry. The world of algebraic geometry is markedly more rigid than symplectic geometry. For instance, the local behaviour of a univariate polynomial function (say, its values on a sufficiently large finite set of points) completely determines its global behaviour.

We stress that this chapter is not intended to be a comprehensive introduction to classical algebraic geometry. Rather, we aim to be minimal in our exposition and focus only on what will be necessary for us in later chapters. Thus, the results in this chapter are stated without proof, but some explanation is occasionally given. For a more thorough exposition of algebraic geometry, with proofs, the reader is encouraged to seek out *Basic Algebraic Geometry* by Shafarevich [SR94], and *Undergraduate Algebraic Geometry* by Reid [Rei88].

In this chapter, we have split classical algebraic geometry into a section on the affine theory and a section on the projective theory. As we mentioned in the introduction, the definitions and theorems covered here will be given over the complex field for simplicity.

3.1 Affine Algebraic Geometry

We begin our journey into algebraic geometry with the study of affine varieties over $\mathbb{C}$, which are spaces cut out by polynomial equations in many variables. In this section we go through the main definitions and results in affine algebraic geometry needed for Chapter 4.

Denoting the set of polynomials in n variables over $\mathbb{C}$ by $\mathbb{C}[x_1, \ldots, x_n]$, we note that $\mathbb{C}[x_1, \ldots, x_n]$ is a $\mathbb{C}$-algebra; that is, a complex vector space with a ring structure. Now take some subset $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$ of polynomials and consider for which points of the affine space[1] $\mathbb{C}^n$ they vanish.

Definition 3.1.1. *An affine variety $V(S) \subseteq \mathbb{C}^n$ is the zero locus of a subset S of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. That is,*

$$V(S) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

One can quickly check that $V(S) = V(\langle S \rangle)$ where $\langle S \rangle$ is the ideal generated by S. As $\mathbb{C}$ is a Noetherian ring, Hilbert's Basis Theorem (Appendix) implies that $\langle S \rangle$ is finite; so one may take the subset of polynomials S to be finite in the definition of an affine variety. If S consists of the polynomials $f_1, \ldots, f_r$, we denote $V(S)$ by $V(f_1, \ldots, f_r)$ for notational convenience.

Although we are working over the field $\mathbb{C}$, it is helpful to visualise the examples in this chapter over the field $\mathbb{R}$.

Example 3.1.2. Consider the affine varieties $V(x)$ and $V(y)$ in $\mathbb{C}^2$ which are simply the x and y axes. Taking their intersection, $V(x) \cap V(y)$, we obtain a single point $(0, 0) \in \mathbb{C}^2$, which is the zero locus of the ideal $\langle x, y \rangle$. On the other hand, the union $V(x) \cup V(y)$ is the zero locus of the ideal $\langle xy \rangle$. So we have $V(x) \cap V(y) = V(x, y)$ and $V(x) \cup V(y) = V(xy)$.

This example suggests at the following properties:

- If I and J are ideals of $\mathbb{C}[x_1, \ldots, x_n]$, then $V(I) \cup V(J) = V(I \cap J)$;
- If $\{I_\alpha\}$ is a collection of ideals of $\mathbb{C}[x_1, \ldots, x_n]$, then $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.

Indeed, these properties turn out to be true of affine varieties in general and allow us to define a topology on $\mathbb{C}^n$ called the *Zariski topology.*

Definition 3.1.3. *We define the* Zariski topology *on $\mathbb{C}^n$ by assigning the closed sets to be the affine varieties in $\mathbb{C}^n$. For a closed subset of $X \subseteq \mathbb{C}^n$, the* Zariski topology *on X is given by the subspace topology.*

We remark that the Zariski topology on $\mathbb{C}^n$ is much coarser than the Euclidean topology on $\mathbb{C}^n$; that is, there are fewer open sets. As an example, the Zariski topology on the affine space $\mathbb{C}$ is the co-finite topology. Recall that a subset of a set X is closed in the co-finite topology if and only if it is finite or X itself. Moreover, the Zariski topology is rarely Hausdorff.

When confusion is likely regarding which topology (Euclidean or Zariski) is being used on $\mathbb{C}^n$, we will make it explicit. For the remainder of this section and the next, however, we will only make use of the Zariski topology.

[1]The terminology 'affine space' will be used throughout this thesis to mean $\mathbb{C}^n$, but keep in mind that it really is not much more than useful terminology for us.

Definition 3.1.4. *An affine variety $X \subseteq \mathbb{C}^n$ is irreducible if it cannot be written as a union of two proper closed subsets of X. An affine variety that is not irreducible is called reducible.*[2]

For example, the affine variety $V(xy)$, from the previous example, is reducible since $V(xy) = V(x) \cup V(y)$.

Affine varieties are geometric objects in the loose sense that they are subsets of the geometric space $\mathbb{C}^n$ and can be drawn or visualised (for small enough dimensions). As in the case of manifold theory, one way to study a geometric object is to look at functions on them; which can be thought of as taking measurements of the object. For a smooth manifold M, we do this by considering the algebra of smooth functions, $C^\infty(M)$, from M to $\mathbb{R}$.

Studying smooth manifolds in this manner suggests an analogous approach for the affine varieties setting. Namely, a suitable way to study affine varieties geometrically is to employ 'algebraic' functions (such as polynomial functions or rational functions), with outputs in the complex numbers.

Suppose that $X \subseteq \mathbb{C}^n$ is an affine variety. We would like to define 'algebraic' functions on X by restricting polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ (thought of as functions $f : \mathbb{C}^n \to \mathbb{C}$) to X. However, two different polynomials p_1 and p_2 in $\mathbb{C}[x_1, \ldots, x_n]$ might be equal after restricting them to X; that is, their difference $p_1 - p_2$ vanishes on X. Thus, we should quotient out by the polynomials that vanish on X.

Definition 3.1.5. *For an affine variety $X \subseteq \mathbb{C}^n$, the set of polynomials which vanish on X is given by*

$$I(X) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

One easily observes that $I(X)$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. By this observation, we make the following definition. First we recall that a ring is nilpotent-free[3] if it has no non-zero nilpotent elements. For our purposes, it will be clear that all rings from here on are nilpotent-free.

Definition 3.1.6. *The coordinate algebra of X is the nilpotent-free, finitely-generated $\mathbb{C}$-algebra*

$$\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/I(X).$$

It is not hard to show that any finitely-generated $\mathbb{C}$-algebra is isomorphic to one of the above form. For instance, let A be a finitely-generated $\mathbb{C}$-algebra and choose generators $a_1, \ldots, a_m \in A$. This defines a surjective $\mathbb{C}$-algebra homomorphism ϕ :

[2] The notion of reducibility given here can be defined for any topological space.

[3] Nilpotent-free rings are also called reduced rings.

$\mathbb{C}[x_1, \ldots, x_m] \to A$ by $x_i \mapsto a_i$ (evaluation map), and so by the first isomorphism theorem, one has $\mathbb{C}[x_1, \ldots, x_m]/\ker \phi \cong A$.

We note that the coordinate algebras of *irreducible* affine varieties have a particularly nice property. First, we remind the reader that a ring is a domain if it has no non-zero zero divisors.

Proposition 3.1.7. *An affine variety $X \subseteq \mathbb{C}^n$ is irreducible if and only if its coordinate algebra $\mathbb{C}[X]$ is a domain.*

Some authors require that affine varieties are irreducible as part of their definition; but for us, affine varieties may be irreducible or reducible.

It turns out that there is a duality between affine varieties and their coordinate algebras, called the *algebra-geometry duality.* Before we can discuss the duality, we need a notion of morphisms between affine varieties.

Definition 3.1.8. *An affine morphism[4] $\varphi : X \to Y$ between affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ is a map of the form*

$$\varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x)) \in Y$$

where $\varphi_i \in \mathbb{C}[X]$ and $x \in X$. If φ is bijective and its inverse is also an affine morphism, then φ is called an affine isomorphism.

For example, the map $\psi : \mathbb{C} \to V(y - x^2) \subseteq \mathbb{C}^2$ defined by $t \mapsto (t, t^2)$ is an affine isomorphism between the affine varieties $\mathbb{C}$ and $V(y - x^2) \subseteq \mathbb{C}^2$. Thus, in this example, we consider $\mathbb{C}$ and X as the same affine variety that is embedded into ambient affine space in two different ways.

Next note that an affine morphism, between two affine varieties, induces a $\mathbb{C}$-algebra homomorphism (a $\mathbb{C}$-linear ring homomorphism) on their corresponding coordinate algebras.

Proposition-Definition 3.1.9. *Let $\varphi : X \to Y$ be an affine morphism between affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$. The pullback of φ,defined by*

$$\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]; \quad f \mapsto f \circ \varphi,$$

is a $\mathbb{C}$-algebra homomorphism.

Following on from the previous example, the coordinate algebras of $\mathbb{C}$ and X are $\mathbb{C}[t]$ and $\mathbb{C}[x, y]/\langle y - x^2\rangle$ respectively. On the generators x and y (mod $\langle y - x^2\rangle$)

[4]Elements of the coordinate algebra of an affine variety are often called regular functions in the standard literature, and affine morphisms as defined here are often called regular maps, although we will not use this terminology.

of $\mathbb{C}[X]$, the pullback $\psi^* : \mathbb{C}[x,y]/\langle y-x^2\rangle \to \mathbb{C}[t]$ is given by $\psi^*(x) = t$, $\psi^*(y) = t^2$. Then, as ψ^* is a $\mathbb{C}$-algebra homomorphism, it is defined on all of $\mathbb{C}[X]$. Note that t and t^2 generate $\mathbb{C}[t]$ and so we have

$$\mathbb{C}[x,y]/\langle y - x^2\rangle \cong \mathbb{C}[t].$$

Thus, the coordinate algebras $\mathbb{C}[X]$ and $\mathbb{C}[t]$ are isomorphic as $\mathbb{C}$-algebras. This example illustrates that different choices of generators for $\mathbb{C}[t]$ correspond to different embeddings of the affine variety $\mathbb{C}$ into ambient affine space.

Moreover, the example above suggests that the geometry of affine varieties is entirely captured in the algebra of its coordinate algebra; and vice versa. Next we make this idea precise as follows.

Note that the affine varieties over $\mathbb{C}$ together with affine morphisms form a category. Furthermore, the nilpotent-free, finitely-generated $\mathbb{C}$-algebras together with $\mathbb{C}$-algebra homomorphisms also form a category. We now state the algebra-geometry duality.

Theorem 3.1.10 (**Algebra-Geometry Duality**). *We define a contravariant functor from the category of affine varieties over $\mathbb{C}$ and the category of nilpotent-free, finitely-generated $\mathbb{C}$-algebras by*

- *associating to each affine variety $X \subseteq \mathbb{C}^n$ its coordinate algebra $\mathbb{C}[X]$;*
- *associating to each affine morphism $\varphi : X \to Y$ its pullback $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$.*

The algebra-geometry duality is the statement that this functor is a duality of categories. That is, for any affine varieties X and Y (over $\mathbb{C}$), the map

$$\left\{\begin{matrix}\text{affine variety morphisms}\\ X \to Y\end{matrix}\right\} \to \left\{\begin{matrix}\mathbb{C}\text{-algebra homomorphisms}\\ \mathbb{C}[Y] \to \mathbb{C}[X]\end{matrix}\right\},$$

given by $\varphi \mapsto \varphi^*$, is a bijection; and each nilpotent-free, finitely-generated $\mathbb{C}$-algebra is isomorphic to $\mathbb{C}[Z]$ for some affine variety Z over $\mathbb{C}$.

We remind the reader that implicit in the definition of a contravariant functor is that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ for all morphisms of affine varieties $\varphi : X \to Y$ and $\psi : Y \to Z$; and that $(\mathrm{id}_X)^* = \mathrm{id}_{\mathbb{C}[X]}$ for any affine variety X.

We now explain in more detail how we get an affine variety from a nilpotent-free, finitely-generated $\mathbb{C}$-algebra as follows. Let A be a nilpotent-free, finitely-generated $\mathbb{C}$-algebra and pick a finite set of generators $a_1, \ldots, a_n$ in A. Then there exists a surjective $\mathbb{C}$-algebra homomorphism

$$\Phi : \mathbb{C}[x_1, \ldots, x_n] \to A$$

such that $\Phi(x_i) = a_i$ for every $i = 1, \ldots, n$. Since $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring, the kernel of Φ is finitely-generated. Thus, $X = V(\ker \Phi)$ is an affine variety contained in $\mathbb{C}^n$. We denote the affine variety X by Spec A[5]. The upshot of all of this is that a choice of generators for A defines an embedding of the affine variety Spec A into an ambient affine space.

Corollary 3.1.11. *Affine varieties $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ are isomorphic if and only if their coordinate algebras $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are isomorphic.*

The Algebra-Geometry Duality allows us to study affine varieties purely by studying algebraic functions on them. In other words, the algebra completely captures the geometry. This in stark contrast with what happens for smooth manifolds; the algebra of smooth functions alone does not capture the geometry of a smooth manifold.

The algebra-geometry duality forms the starting point for Grothendieck's definition of an affine scheme associated to any commutative ring. Grothendieck's view was that any commutative ring has a geometric interpretation that can be arrived at by considering the set of prime ideals of the ring, called the spectrum of the ring. This viewpoint is what instigated much of modern algebraic geometry.

We conclude this section with a nice result about the open subsets of an affine variety. First we need another definition. Suppose $X \subseteq \mathbb{C}^n$ is an affine variety.

Definition 3.1.12. *For any $f \in \mathbb{C}[X]$, the open set*

$$X_f := X \setminus V(f) = \{x \in X \mid f(x) \neq 0\}$$

is called a distinguished open set of X.

The significance of the distinguished open sets of X is two-fold. Firstly, they form a basis for the Zariski topology on X; that is, every open subset of X is a union of distinguished open sets of X. Secondly, distinguished open sets are isomorphic to affine varieties. Putting this together, we have that every open subset of an affine variety is a union of open subsets which are themselves affine varieties.

In the next section, we move on from affine varieties to the more nuanced projective varieties within algebraic geometry.

[5] Abstractly speaking, Spec A is the set of prime ideals of A. See [SR94] for a friendly introduction.

3.2 Projective Algebraic Geometry

In this section, we look at projective varieties, which are slightly more complicated but are often a better setting in which to do algebraic geometry, compared with affine varieties. Part of the reason for this is that parallel lines can meet at a 'point at infinity'. Another reason is that projective varieties are compactifications of affine varieties, yielding a preferable space in which to perform calculations.

The main idea in the construction of projective varieties is as follows. We would like to define an 'abstract' variety (not necessarily affine) by gluing affine varieties together in a suitable way. For our purposes, the most important examples of such objects are complex projective varieties. Hence, in this section, we introduce only the most important and needed definitions and theorems from projective algebraic geometry.

We remind the reader that our definition of complex projective space $\mathbb{CP}^n$ is the set of equivalence classes of points in $\mathbb{C}^{n+1} \setminus \{0\}$, where two points z and w are defined to be equivalent if there exists a non-zero scalar $\lambda \in \mathbb{C}^*$ such that $v = \lambda w$. In other words, the projective space $\mathbb{CP}^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$ (one-dimensional subspaces)[6].

Recall that homogeneous coordinates for $\mathbb{CP}^n$ are denoted $[x_0 : \cdots : x_n] \in \mathbb{CP}^n$, where $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$.

Definition 3.2.1. *A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous of degree d if*

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$$

for every $\lambda \in \mathbb{C}^\times$.

For example, the polynomial $f(x, y) = xy + x^2$ in $\mathbb{C}[x, y]$ is homogeneous of degree 2.

If we try, näively, to define a projective variety in the same manner used for the affine case (cut out by polynomial equations), we may notice that a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is not well-defined *as a function* on $\mathbb{CP}^n$. The polynomial f is not well-defined since points in $\mathbb{CP}^n$ are equivalence classes of points in $\mathbb{C}^{n+1}$, and f need not be constant on these equivalence classes. However, the condition $f(x) = 0$ for $x \in \mathbb{CP}^n$ does make sense if f is a homogeneous polynomial. For if $x = [x_0 : \cdots : x_n]$ is a point in $\mathbb{CP}^n$ with representative $(x_0, \ldots, x_n)$ in $\mathbb{C}^{n+1} \setminus \{0\}$, and $f(x_0, \ldots, x_n) = 0$, then $f(\lambda x_0, \ldots, \lambda x_n) = 0$ for any $\lambda \in \mathbb{C}^\times$ as f is homogeneous. With this in mind, we make the following definition of a projective variety

[6]We remark that our definition of projective space is the differential geometers definition. In modern algebraic geometry, one actually defines it to be the set of hyperplanes through the origin (codimension-1 subspaces).

contained in projective space.

Definition 3.2.2. *A projective variety $V(S) \subseteq \mathbb{CP}^n$ is the zero locus of a subset S of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. That is,*

$$V(S) = \{x \in \mathbb{CP}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

Similarly to the affine case, we have $V(S) = V(\langle S \rangle)$ so it suffices to consider $V(I)$ where I is a homogeneous ideal (an ideal generated by homogeneous polynomials) of $\mathbb{C}[x_0, \ldots, x_n]$.

For example, the ideal $I = \langle x_0, \ldots, x_n \rangle$ is a homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_n]$ and the affine variety in $\mathbb{C}^{n+1}$ defined by I is the point $\{0\}$, which corresponds to the empty set in $\mathbb{CP}^n$. As such, we call this ideal the *irrelevant ideal*.

It is important to note that our definition of projective varieties assumes a given embedding into projective space (through set containment). This is similar to our definition of affine varieties, however what follows depends heavily on the projective variety embedding. Abstractly, one instead defines a projective variety to include the data of a *line bundle*, which then defines an embedding into ambient projective space. Loosely speaking, a line bundle over a projective variety is a collection of one-dimensional vector spaces that are parameterised by the points of the variety[7].

The Zariski topology for $\mathbb{CP}^n$ is defined analogously to the affine case, with projective varieties in $\mathbb{CP}^n$ forming the closed subsets. For a projective variety $X \subseteq \mathbb{CP}^n$, the Zariski topology on X is defined by the subspace topology inherited from $\mathbb{CP}^n$.

Irreducible and reducible projective varieties contained in $\mathbb{CP}^n$ are also defined analogously to the affine case.

Furthermore, the Zariski topology for a projective variety has a nice basis of *distinguished open sets*, similar to the affine case, that are defined analogously. Distinguished open subsets of a projective variety are also isomorphic to affine varieties, like in the affine case.

To study functions on our projective varieties, we would like an analogue of the coordinate algebra of affine varieties. Let $X \subseteq \mathbb{CP}^n$ be a projective variety.

[7] This is similar to the construction of the tangent and cotangent bundles for smooth manifolds. In fact, all of these objects are examples of what are known as vector bundles. See Lee [Lee12] or Shafarevich [SR94] for an introduction to vector bundles.

Proposition-Definition 3.2.3. *The set of polynomials vanishing on $X \subseteq \mathbb{CP}^n$, given by*
$$I(X) := \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\},$$
is a homogeneous ideal of $\mathbb{C}[x_0, \dots, x_n]$.

Recall that $\mathbb{C}$-algebra A is graded if $A = \bigoplus_{d \geq 0} A_d$, where each A_d is a complex vector space and $A_d A_{d'} \subseteq A_{dd'}$.

Proposition-Definition 3.2.4. *The homogeneous coordinate algebra*[8] *of $X \subseteq \mathbb{CP}^n$ is the nilpotent-free, finitely-generated $\mathbb{C}$-algebra*

$$R(X) := \mathbb{C}[x_0, \dots, x_n]/I(X).$$

Moreover, $R(X)$ is a graded $\mathbb{C}$-algebra:

$$R(X) = \bigoplus_{d \geq 0} (R(X))_d$$

where $(R(X))_d$ are the homogeneous polynomials of degree d (modulo $I(X)$), and $(R(X))_+ := \bigoplus_{d>0} (R(X))_d$ is the irrelevant ideal.

In the construction of the complex projective space $\mathbb{CP}^n$, the affine variety $\mathbb{C}^{n+1}$ is called the affine cone over $\mathbb{CP}^n$. Similarly, given a projective variety $X = V(I) \subseteq \mathbb{CP}^n$ where I is a homogeneous ideal, the *affine variety* defined by I is called the affine cone over X and is denoted by $\widetilde{X}$.

The reason for the name 'affine cone' is that, if a point $(x_0, \dots, x_n)$ belongs to the affine cone $\widetilde{X}$ of a projective variety $X \subseteq \mathbb{CP}^n$, then so does $(tx_0, \dots, tx_n)$ for any $t \in \mathbb{C}^*$. In other words, affine cones contain (some) lines through the origin. For instance, consider the projective variety $X = V(x^2 + y^2 - z^2) \subseteq \mathbb{CP}^2$; that is, the zero locus of the homogeneous polynomial $x^2 + y^2 - z^2$ in the projective plane. The affine cone is defined by the same polynomial in the complex affine space $\mathbb{C}^3$, and is an honest cone with circular cross-sections if one visualises its real locus.

A useful characterisation of complex projective varieties is that we can obtain them by gluing together affine varieties. For instance, consider the $(n + 1)$ open subsets of complex projective space $\mathbb{CP}^n$ given by

$$U_i := \{[x_0 : \dots : x_n] \in \mathbb{CP}^n \mid x_i \neq 0\}$$

[8]The definition of the homogeneous coordinate algebra of a projective variety given here depends on an embedding into projective space. However, since we are assuming that our projective varieties are already contained in projective space, this will not be of any concern to us.

for each $i = 0, \ldots, n$. It is clear that their union is $\mathbb{CP}^n$, so these sets cover projective space. We next show that the U_i are also affine varieties. The maps $\varphi_i : U_i \to \mathbb{C}^n$ defined by

$$[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right)$$

are isomorphisms, with inverses $\varphi_i^{-1} : \mathbb{C}^n \to U_i$ given by

$$(x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n].$$

Note that the coordinate algebra of each U_i is

$$\mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \cong \mathbb{C}[y_1^{(i)}, \ldots, y_n^{(i)}].$$

We call the open sets U_i the standard affine patches of $\mathbb{CP}^n$ since, by the above, they are each isomorphic to copies of the affine space $\mathbb{C}^n$.

As an example, the complex projective line (Riemann sphere) $\mathbb{CP}^1$ is glued together by two affine planes $U_0 \cong \mathbb{C}^1_{x_1/x_0}$ and $U_1 \cong \mathbb{C}^1_{x_0/x_1}$. To somewhat visualise, we can picture the following process. First, we wrap each plane into a punctured sphere. We orient one of the spheres so that it is punctured at its north pole, and we orient the other sphere so that it is punctured at its south pole. The punctured pole on each sphere represents a missing 'point at infinity' for each affine plane, while the opposing poles represent the origin of each affine plane. Now to obtain the complex projective line $\mathbb{CP}^1$, we glue these punctured spheres together by identifying their north poles and identifying their south poles, as can be seen in Figure 3.1.

Similarly, any projective variety $X \subseteq \mathbb{CP}^n$ is covered by $(n+1)$ affine open subsets $X \cap U_i$ where the U_i are defined as above. In essence, this notion of constructing projective varieties by gluing together affine varieties, mirrors the philosophy seen in differential geometry. The philosophy seen in differential geometry is that manifolds are glued together by open subsets that are homeomorphic to Euclidean space.

We next look at *regular functions* defined on open subsets of projective varieties, which may be thought of as rational functions that are partially defined. First, we remind the reader that a polynomial function $f \in \mathbb{C}[x_0, \ldots, x_n]$ is not well-defined as a function on a complex projective variety $X \subseteq \mathbb{CP}^n$. Nonetheless, the assertion $f(x) = 0$, for $x \in X$, makes sense as long as f is homogeneous. For non-zero values of f, the only homogeneous polynomials that are well-defined on X are the constant ones, which do not give much information about X.

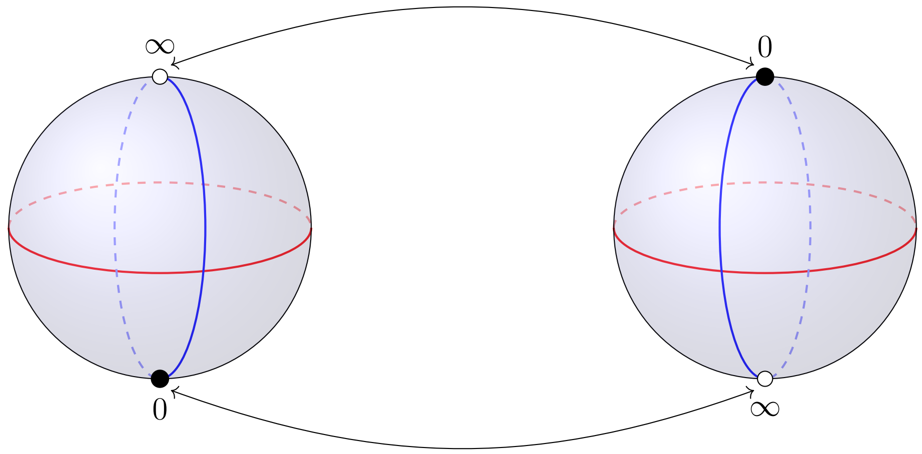


Figure 3.1: Complex projective line obtained by gluing two complex planes.

Instead, we consider so-called *regular functions* on X. For completeness of our exposition of classical algebraic geometry, we define *regular functions* on open subsets of *affine* or *projective* varieties. Suppose U is an open subset of an affine variety $X \subseteq \mathbb{C}^n$.

Definition 3.2.5. *A function $f : U \to \mathbb{C}$ is* regular *at $x \in U$ if it can be expressed in the form $f = P/Q$ for some $P, Q \in \mathbb{C}[X]$ and $Q(x) \neq 0$. If f is regular at every $x \in U$ then f is called a* regular function. *We denote the $\mathbb{C}$-algebra of regular functions on U by $\mathcal{O}_X(U)$.*

As an example, the rational function $1/x$ is regular on the open subset $\mathbb{C}^1 \setminus V(x)$ of $\mathbb{C}^1$.

It turns out that, for an *irreducible* affine variety, its $\mathbb{C}$-algebra of regular functions is isomorphic to its coordinate algebra. A proof of this fact may be found in [SR94, Theorem 1.7].

In the projective case, one needs to be careful to ensure that a quotient of polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ is well-defined *as a function* on $\mathbb{CP}^n$. That is, its values should not depend on choices of equivalence class representatives in $\mathbb{C}^{n+1}$. Thus, any such quotient must be given by homogeneous polynomials of the *same degree*, since

$$\frac{P(\lambda x_0, \ldots, \lambda x_n)}{Q(\lambda x_0, \ldots, \lambda x_n)} = \frac{\lambda^d P(x_0, \ldots, x_n)}{\lambda^d Q(x_0, \ldots, x_n)} = \frac{P(x_0, \ldots, x_n)}{Q(x_0, \ldots, x_n)}$$

where $P, Q \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials of degree d. With this additional subtlety, the definition of regular functions in the projective case mirrors that of regular functions in the affine case. Suppose U is an open subset of a projective variety $X \subseteq \mathbb{CP}^n$.

Definition 3.2.6. *A function $f : U \to \mathbb{C}$ is* regular *at $x \in U$ if it can be expressed in the form $f = P/Q$ for some homogeneous $P, Q \in R(X)_d$ of the same degree, and $Q(x) \neq 0$. If f is regular for all $x \in U$, then f is called a* regular function*. We denote the $\mathbb{C}$-algebra of regular functions on U by $\mathcal{O}_X(U)$.*

We conclude our discussion of projective algebraic geometry with a definition of morphisms of projective varieties. Suppose $X \subseteq \mathbb{CP}^n$ and $Y \subseteq \mathbb{CP}^m$ are projective varieties.

Definition 3.2.7. *A projective morphism (or rational morphism)*[9] *is a partially defined map $\varphi : X \dashrightarrow Y$ of the form*

$$\varphi(x) = [\varphi_1(x) : \cdots : \varphi_m(x)] \in Y$$

where $\varphi_1, \ldots, \varphi_m$ are rational functions $\varphi_i = P_i/Q_i$ and $P_i, Q_i \in R(X)_d$. If each φ_i is regular at $x \in X$ and not all zero at x, then φ is well-defined at x.

If φ has an inverse which is also a projective morphism, then φ is called a projective isomorphism, or an isomorphism of projective varieties.

Upon multiplying by a common denominator, a projective morphism may equivalently be defined by

$$\varphi : X \dashrightarrow Y, \;\; x \mapsto [F_0(x) : \cdots : F_m(x)],$$

where $F_0, \ldots, F_m \in R(X)_d$ and $x \in X \setminus V(F_0, \ldots, F_m)$.

For an example of a projective morphism, consider the map $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^2$ given by

$$[x : y] \mapsto [x^2 : xy : y^2] = \left[\left(\frac{x}{y}\right)^2 : \frac{x}{y} : 1 \right] = \left[1 : \frac{y}{x} : \left(\frac{y}{x}\right)^2 \right],$$

where $\mathrm{Im}(\varphi) = V(x_0x_2 - x_1^2)$. Note that this map is well-defined as $[\lambda x : \lambda y] \mapsto [\lambda^2 x^2 : \lambda^2 xy : \lambda^2 y^2] = [x^2 : xy : y^2]$ and as a map onto its image, φ has an inverse $\varphi^{-1} : \mathrm{Im}(\varphi) \to \mathbb{CP}^1$ given by

$$[x_0 : x_1 : x_2] \mapsto [x_0 : x_1] = [x_1 : x_2]$$

[9] We will use these terms interchangeably.

as $x_0x_2 = x_1^2$ on $\mathrm{Im}(\varphi)$. Thus, φ is an isomorphism of $\mathbb{CP}^1$ and the curve[10] $X \subseteq \mathbb{CP}^2$ cut out by the polynomial equation $x_0x_2 = x_1^2$. On the affine patch $x_0 \neq 0$, we can see that X is a parabola.

The reader should note that the algebra-geometry duality does not exist for projective varieties. The reason for this is that the homogeneous coordinate algebra of a projective variety heavily depends on an embedding of the projective variety into ambient projective space, as mentioned earlier. For instance, the map in the above example defines an embedding of $\mathbb{CP}^1$ into $\mathbb{CP}^2$. The homogeneous coordinate algebra for this particular embedding is given by

$$\mathbb{C}[x_0, x_1, x_2]/\langle x_0x_2 - x_1^2\rangle,$$

which is not a UFD (unique factorisation domain) as $x_1^2 = x_0x_2$ are two different factorisations of the same element. On the other hand, $\mathbb{C}[x, y] = R(\mathbb{CP}^1)$ is a UFD, and so these rings are not isomorphic.

Now that we have covered the basics of algebraic geometry, we are ready to progress onward to investigating group actions on our algebraic varieties, resulting in Mumford's *Geometric Invariant Theory*!

[10]This curve is called the rational normal curve or Veronese variety.

Chapter 4

Quotients in Algebraic Geometry: Geometric Invariant Theory

"Like the Arabian phoenix rising out of its ashes, the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics. During its long eclipse, the language of modern algebra was developed, a sharp tool now at last being applied to the very purpose for which it was intended."
– Kung and Rota, *The invariant theory of binary forms*, 1984 [KR84].

In this chapter, we investigate group actions on algebraic varieties and the quotients that arise from them. Much like with group actions in symplectic geometry, the theory of quotients in algebraic geometry is also rather subtle. So much so in fact, that it was named Geometric Invariant Theory (GIT) by Mumford. Mumford invented GIT in the 1960s, after excavating and using ideas from a widely unknown 1893 paper of Hilbert on classical invariant theory. Mumford's work on GIT is given in his book *Geometric Invariant Theory* [MFK94], which is one of the main references for this chapter.

We importantly comment that our treatment of GIT given in this chapter will not include proofs, similar to the previous chapter. We skip proofs because they require more technicality and definitions than we have space to present here, so we encourage the reader to see Mumford's book on GIT [MFK94], Hoskins' notes [Hos12], or the lecture notes by Newstead [New06] for a more detailed exposition.

However, we have attempted to illustrate why certain results are true with detailed and worked through examples, as well as diagrams and pictures to go along with the explanations.

Section 4.1 covers the affine case of GIT, which looks at group actions on *affine* varieties. While Section 4.2 covers the more demanding projective case of GIT, which inspects group actions on *projective* varieties.

4.1 Affine Geometric Invariant Theory

In this section, we briefly explore group actions on affine varieties with the aim of defining the *affine GIT quotient*. As we are dealing with algebraic objects, we need a notion of an *algebraic* group that acts on algebraic varieties. Thus, we begin with the following definition.

Definition 4.1.1. *An affine algebraic group G (over $\mathbb{C}$) is an affine variety (over $\mathbb{C}$) such that the multiplication and inversion maps are affine morphisms.*

The reader should note the definition given here is analogous to our definition of a Lie group. We could have instead defined algebraic groups to be Zariski-closed subsets of $\mathrm{GL}_n(\mathbb{C})$, but we have chosen our definition to emphasise the analogy with differential geometry.[1] Some examples of affine algebraic groups include $\mathrm{GL}_n(\mathbb{C})$ and $\mathbb{C}^*$.

Given an affine variety X and an affine algebraic group G, we consider group actions of G on X that are affine morphisms. If that is the case, we call the pair (X, G) together with such action an *affine G-variety.*

We would like a suitable definition for a quotient of an affine variety by an affine algebraic group. However, we will see that the orbit space quotient does not suffice. The reasons why are subtle. By example we illustrate where things go wrong.

Example 4.1.2. Consider the action of $G = \mathbb{C}^*$ on $X = \mathbb{C}^2$ given by $\lambda \cdot (x, y) := (\lambda^{-1}x, \lambda y)$. There are three kinds of orbits:

- Hyperbolae $H_\alpha = \{(x, y) \in \mathbb{C}^2 \mid xy = \alpha\}$ for $\alpha \in \mathbb{C} \setminus \{0\}$;
- Punctured x and y axes $\{(x, 0) \in \mathbb{C}^2 \mid x \neq 0\}$, $\{(0, y) \in \mathbb{C}^2 \mid y \neq 0\}$;
- Origin $\{(0, 0)\}$.

We observe that the hyperbolae are closed orbits (in the Zariski and Euclidean topologies), and the punctured x and y axes are non-closed orbits containing the origin in their closure. The orbit space is thus not Hausdorff (in the Euclidean topology). While, if we remove the origin, which is a zero-dimensional orbit, the limit as α tends to zero, of the hyperbolae H_α, are both the punctured x and y axes. Thus, the resulting orbit space is still not Hausdorff.

The above example highlights the fact that the usual topological quotient (orbit space) does not inherit properties (such as being Hausdorff) from the original space. We would of course like such inheritance to occur. For instance, it is not an affine variety, which is not desirable.

[1]The alternative definition is commonly known as the definition of a linear algebraic group. It turns out that these two definitions are equivalent.

Our approach is to relax what is meant by quotient here. The Algebra-Geometry Duality serves as inspiration for the approach we will take. We remind the reader that the Algebra-Geometry Duality says there is a bijection between affine varieties over $\mathbb{C}$ and finitely-generated nilpotent-free $\mathbb{C}$-algebras.

With this in mind, the idea is to define the affine GIT quotient, of an affine variety by an affine algebraic group, by taking the spectrum of a suitable $\mathbb{C}$-algebra that captures information about how the group acts on the coordinate algebra of the variety.

Suppose that X is an affine G-variety and assume that Y is an affine variety representing the 'affine GIT quotient of X by G'. Then a 'quotient morphism' $\varphi : X \to Y$ should be a surjective affine morphism. Assuming this, then by the Algebra-Geometry Duality, the $\mathbb{C}$-algebra homomorphism $\varphi^* : \mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]$ is injective. Thus, we can consider $\mathbb{C}[Y]$ as a subalgebra of $\mathbb{C}[X]$.

Furthermore, φ should be G-invariant (constant on orbits). In other words, we should have

$$\varphi \circ \sigma_g = \varphi,$$

where $\sigma_g : X \to X$ $(x \mapsto g \cdot x)$, for every $g \in G$. Upon taking pullbacks, this condition is equivalent to the condition that

$$\sigma_g^* \circ \varphi^* = \varphi^*$$

on the dual algebraic side.

Definition 4.1.3. *Let $X \subseteq \mathbb{C}^n$ be an affine G-variety. The action of G on X induces an action on the coordinate algebra $\mathbb{C}[X]$ by*

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

for $g \in G$, $f \in \mathbb{C}[X]$, and $x \in X$. The ring of invariants is the $\mathbb{C}$-subalgebra of $\mathbb{C}[X]$ given by

$$\mathbb{C}[X]^G := \{f \in \mathbb{C}[X] \mid g \cdot f = f \text{ for all } g \in G\}.$$

Thus, $\mathbb{C}[Y]$ is a subalgebra of $\mathbb{C}[X]$ subject to the condition

$$\sigma_g^* \mathbb{C}[Y] \subseteq \mathbb{C}[X]^G, \tag{4.1.1}$$

for all $g \in G$. We would prefer to avoid identifying points in the same orbit any further than necessary. This preference corresponds to taking the largest subalgebra of $\mathbb{C}[X]^G$ such that (4.1.1) holds; which is, of course, the algebra of invariants $\mathbb{C}[X]^G$ itself.

We would like to define the (affine) GIT quotient of X by G to be the pair (Y, φ) where $\varphi : X \to Y$ is the affine morphism dual to the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. However, this is contingent on the assumption of $\mathbb{C}[X]^G$ being finitely-generated, an assumption which is necessary in order to apply the algebra-geometry duality at all (the fact that $\mathbb{C}[X]^G$ is nilpotent-free is clear). As it turns out, the case of being finitely-generated is only true for certain classes of affine algebraic groups called reductive groups[2].

Definition 4.1.4. *An affine algebraic group G is called reductive if every linear representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is completely reducible; that is, ρ splits into a direct sum of irreducible representations.*

Some standard examples of reductive groups to keep in mind include $(\mathbb{C}^*)^n$, $\mathrm{GL}_n(\mathbb{C})$, and $\mathrm{SL}_n(\mathbb{C})$, which are our main working examples.

We remark that the definition of reductivity given here is one of many, all of which turn out to be equivalent over $\mathbb{C}$. This equivalence is false over arbitrary fields.[3] Nevertheless, as we are only working over $\mathbb{C}$, we will stick with the above definition. From now on, when we say affine G-variety, the affine algebraic group G is assumed to be reductive.

Let us now define the (affine) GIT quotient. Suppose X is an affine G-variety.

Definition 4.1.5. *The* (affine) GIT quotient *is the pair (Y, φ), where*

$$\varphi : X \to Y := \operatorname{Spec} \mathbb{C}[X]^G$$

is the affine morphism induced by the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$.

Unfortunately, unlike the orbit space, the affine GIT quotient may not separate orbits. However, it does exhibit some of the following nice properties[4]:

- The map $\varphi : X \to Y$ is G-invariant and surjective;
- If $U \subseteq Y$ is Zariski-open then the ring homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))^G$ is an isomorphism;
- If W_1, W_2 are disjoint Zariski-closed G-invariant subsets of X then $\varphi(W_1)$, $\varphi(W_2)$ are disjoint Zariski-closed subsets of Y.

Now recall Example 4.1.2. With our newly available tools, let us calculate its affine GIT quotient.

[2]A version of Hilbert's 14th problem asks whether or not the ring of invariants $\mathbb{C}[X]^G$ is always finitely-generated, which was answered in the negative by Nagata (see [Nag+63] &[Nag59]).

[3]See the notes by Hoskins [Hos12, p. 14].

[4]These properties form part of the definition of a 'good' quotient (see Hoskins' notes [Hos12, p. 11]). So what these properties say is that the affine GIT quotient is a 'good' quotient.

Example 4.1.6. For the action of $G = \mathbb{C}^*$ on $X = \mathbb{C}^2_{x,y}$ given by $\lambda \cdot (x,y) = (\lambda^{-1}x, \lambda y)$, we first calculate the ring of invariants $\mathbb{C}[x,y]^{\mathbb{C}^*}$. Let $\lambda^* : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ denote the pullback of the action of $\lambda \in \mathbb{C}^*$ on $\mathbb{C}^2$. It suffices to compute on the monomials $x^i y^j$ since λ^* is a $\mathbb{C}$-algebra homomorphism and we have the direct sum $\mathbb{C}[x,y] = \bigoplus_{i,j \geqslant 0} \mathbb{C}x^i y^j$. Thus, we compute

$$\lambda^*(x^i y^j) = ((\lambda^{-1})^i x^i)(\lambda^j y^j) = \lambda^{-i+j} x^i y^j$$

so we see that $\lambda^*(x^i y^j) = x^i y^j$ if and only if $i = j$. Hence, the ring of invariants is given by

$$\mathbb{C}[x,y]^{\mathbb{C}^*} = \bigoplus_{i \geqslant 0} \mathbb{C}(xy)^i = \mathbb{C}[xy] \cong \mathbb{C}[z]$$

and the affine GIT quotient for this action is simply the pair $(\mathbb{C}, \varphi)$, where

$$\varphi : \mathbb{C}^2_{x,y} \mapsto \mathbb{C}_{xy} \cong \mathbb{C}_z, \ (x,y) \mapsto xy.$$

Note that the preimage $\varphi^{-1}(0)$ is the union of the punctured (at the origin) x and y axes, which are non-closed orbits, and the origin itself, which is a closed orbit. So in particular, the affine GIT quotient is not the same as the orbit space as we may have expected.

Having dealt with some of the affine component of GIT, we now move onto the more nuanced projective case.

4.2 Projective Geometric Invariant Theory

In this section, we explore projective GIT, which concerns itself with reductive group actions on projective varieties. Given an appropriate action of a reductive group on a complex projective variety, the aim of this section is to define the *projective GIT quotient* of the projective variety by the group. The situation here is more subtle than in the affine case, but the main idea in constructing the projective GIT quotient is to glue together affine GIT quotients. We begin with the definition of a *linear* action of a reductive group on a projective variety. Suppose $X \subseteq \mathbb{CP}^n$ is a complex projective variety and let G be a reductive group.

Definition 4.2.1. *A* linear *action of G on X is given by a linear representation $\rho : G \to \mathrm{GL}_{n+1}(\mathbb{C})$.*

Similarly with the affine case, we call such pairs $(X \subseteq \mathbb{CP}^n, G)$ *projective G-varieties.*

We remind the reader that our definition of a projective variety includes an embedding into projective space (through set containment). In contrast to the affine case, we would expect that the projective GIT quotient of a projective G-variety, which we soon define, will depend on its embedding into projective space.

Due to this extra complication, a G-action on X depends on a linearisation of the action; that is, we need to specify a lift, of the G-action to the line bundle over X, of which there may be many.

For us however, we assume that all our projective varieties are already embedded in $\mathbb{CP}^n$, and furthermore that G acts through $\mathrm{GL}_{n+1}(\mathbb{C})$, so G also acts on the $\mathbb{C}^{n+1}$ and the affine cone $\widetilde{X}$ over X. This is precisely a choice of a linearisation of the action.

To reiterate, the key idea to obtaining the projective GIT quotient is to glue together affine GIT quotients; that is, we decompose X into a union of G-invariant affine open subsets, take the (affine) GIT quotients of each of these subsets, and then glue them together. The subtlety here is that such a decomposition of X in general only works for a Zariski-open subvariety of X called the semistable set.

Recall that the homogeneous coordinate algebra (which is graded by homogeneous degree) of X and the irrelevant ideal are given respectively by

$$R(X) = \mathbb{C}[x_0, \dots, x_n]/I(X) = \bigoplus_{d \geqslant 0} R(X)_d, \quad R(X)_+ = \bigoplus_{d > 0} R(X)_d.$$

The action of G on $X \subseteq \mathbb{CP}^n$ lifts to an action on the affine cone $\widetilde{X} \subseteq \mathbb{C}^{n+1}$ and as in the affine case, this induces an action of G on $R(X) = \mathbb{C}[\widetilde{X}]$. As G acts on the affine cone $\widetilde{X}$, it follows that G preserves the graded pieces of $R(X)$. Thus, the ring of invariants is itself a graded (by homogeneous degree) algebra

$$R(X)^G = \bigoplus_{d \geqslant 0} R(X)_d^G.$$

Since G is reductive, $R(X)^G$ is finitely-generated, so the inclusion $R(X)^G \hookrightarrow R(X)$ induces a rational morphism

$$\varphi : X \dashrightarrow X /\!\!/ G := \mathrm{Proj} R(X)^G.$$

This morphism is undefined on the Zariski-closed subset $V(R(X)_+^G)$ since the irrelevant ideal $R(X)_+^G$ contains the ideal $\langle x_0, \dots, x_n \rangle$, which corresponds to the empty set in $\mathbb{CP}^n$.

Definition 4.2.2. *A point $x \in X$ is called*

- semistable[5] *if there exists a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $f(x) \neq 0$;*
- polystable *if it is semistable and the orbit $G \cdot x$ is Zariski-closed;*
- stable *if it is polystable and the stabiliser G_x is finite;*
- unstable *if it is not semistable.*

We denote the set of semistable, polystable, stable and unstable points by X^{ss}, X^{ps}, X^s and X^{us} respectively. Note that we have the containment relations

$$X \supseteq X^{ss} \supseteq X^{ps} \supseteq X^s.$$

Furthermore, we have $X^{ss} = \bigcup_{f \in R(X)^G_+} X_f$, where $X_f = \{x \in X \mid f(x) \neq 0\}$ are affine Zariski-open subsets of X. As f is G-invariant, it follows that X_f is G-invariant:

$$f(g \cdot x) = f(x) \neq 0$$

for any $g \in G$ and $x \in X_f$; thus X^{ss} is also G-invariant. Moreover, observe that $X^{ss} = X \setminus V(R(X)^G_+)$, so in particular X^{ss} is Zariski-open. Thus, the above rational morphism induced by the inclusion $R(X)^G \hookrightarrow R(X)$ restricts to a (well-defined) projective morphism on the semistable subset X^{ss}. Now we may define the projective GIT quotient.

Definition 4.2.3. *Let $X \subseteq \mathbb{CP}^n$ be a projective G-variety, where G is a reductive group. The (projective)* GIT *quotient is the pair $(X /\!\!/ G, \varphi)$ where*

$$\varphi : X^{ss} \to X /\!\!/ G := \operatorname{Proj} R(X)^G$$

is the projective morphism induced by the inclusion of homogeneous coordinate algebras $R(X)^G \hookrightarrow R(X)$.

We remark that the projective GIT quotient satisfies the same properties as the affine GIT quotient that were listed above; for instance, φ is G-invariant (constant on orbits) and surjective. Perhaps the most important property of the projective GIT quotient is that, it is itself a complex projective variety. That is, the projective GIT quotient can be embedded into complex projective space.

We may consider the following as the main result of GIT.

Theorem 4.2.4 (Mumford [MFK94])**.** *Let $X \subseteq \mathbb{CP}^n$ be a projective G-variety. Then the projective GIT quotient $(X /\!\!/ G, \varphi)$ has the following properties:*

1. *φ restricts to a (geometric) quotient[6] on the stable set $X^s \subseteq X$.*

[5] The definition of a semistable point is algebraic in the sense that they are the points which are 'detected' by the G-invariant functions on X.

[6] That is, $\varphi(X^s)$ is an orbit space, so distinct orbits are separated.

2. $X /\!\!/ G$ is in bijective correspondence with the closed orbits in X^{ss}; in other words, $X /\!\!/ G \cong_{set} X^{ps}/G$, where $X^{ps} \subseteq X^{ss}$ is the set of polystable points.

As the beginning of this section mentioned, the projective GIT quotient can also be obtained by gluing affine GIT quotients together. The main idea is as follows (full details can be found in [MFK94]). Recalling the decomposition of the semistable set

$$X^{ss} = \bigcup_{f \in R(X)_+^G} X_f,$$

the reductivity of G implies that $R(X)^G$ is finitely-generated as a $\mathbb{C}$-algebra, so choose G-invariant homogeneous generators $f_1, \ldots, f_m$ which necessarily have degree at least one. Then the decomposition for X^{ss} above can be refined into the union

$$X^{ss} = \bigcup_{i=1}^{m} X_{f_i}.$$

Note that each X_{f_i} is a G-invariant affine open subset and so, in particular, we have (affine) GIT quotients (Y_i, φ_i) given by

$$\varphi_i : X_{f_i} \to Y_i := \operatorname{Spec} \mathbb{C}[X_{f_i}]^G.$$

We can then glue these affine morphisms (by the gluing lemma (Appendix))) in the Zariski topology to obtain a projective morphism

$$\varphi : \bigcup_{i=1}^{m} X_{f_i} \to \bigcup_{i=1}^{m} Y_i =: X /\!\!/ G$$

where $\varphi|_{X_{f_i}} = \varphi_i$.

We now give an example of the projective GIT quotient and also illustrate the gluing process discussed above in this following example.

Example 4.2.5. The first part of this example is a simplified version of an example given by Hoskins in [Hos12], with a few more details added. There they consider $\mathbb{CP}^n$ for n arbitrary, while we just take $n = 2$ here. This was done to simplify the exposition and with consideration for the second part of this example, we can draw pictures more easily.

The second part of the example was heavily inspired by one of Daniel Chan's videos on his Youtube channel 'DanielChanMaths'. In particular, the diagrams/pictures are the author's own.

Consider the action[7] of $G = \mathbb{C}^*$ on $X = \mathbb{CP}^2$ given by

$$\lambda \cdot [x : y : z] := [\lambda^{-1}x : \lambda y : \lambda z],$$

where we lift[8] the action according to $\lambda \cdot (x, y, z) = (\lambda^{-1}x, \lambda y, \lambda z)$ on the affine cone $\mathbb{C}^3$. To find the ring of invariants of $R(\mathbb{CP}^2) = \mathbb{C}[x, y, z]$, it suffices to compute the action on monomials $x^i y^j z^k$ since the action is a $\mathbb{C}$-algebra homomorphism and the homogeneous coordinate algebra is

$$R(\mathbb{CP}^2) = \bigoplus_{i,j,k \geqslant 0} \mathbb{C}x^i y^j z^k.$$

Now, upon computing

$$\lambda \cdot (x^i y^j z^k) = \lambda^{i-j-k} x^i y^j z^k,$$

we see that this monomial is $\mathbb{C}^*$-invariant if and only if $i = j + k$; in which case, we have $x^i y^j z^k = (xy)^j (xz)^k$. Hence, the ring of invariants is given by

$$R(\mathbb{CP}^2)^{\mathbb{C}^*} = \mathbb{C}[xy, xz] \cong \mathbb{C}[a, b]$$

and the inclusion $\mathbb{C}[xy, xz] \hookrightarrow \mathbb{C}[x, y, z]$ induces a rational morphism

$$\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 /\!\!/ \mathbb{C}^* = \mathbb{CP}^1, \ [x : y : z] \mapsto [xy : xz].$$

This rational morphism is undefined when $xy = xz = 0$, or in other words, $V(xy, xz)$ and φ restricts to a well-defined projective morphism on the semistable set

$$(\mathbb{CP}^2)^{ss} = \mathbb{CP}^2 \setminus V(xy, xz) = \{[x : y : z] \in \mathbb{CP}^2 \mid x \neq 0 \text{ and } (y, z) \neq 0\} \cong \mathbb{C}^2 \setminus \{0\}$$

where we have identified the affine patch $\mathbb{CP}^2_{x \neq 0}$ with $\mathbb{C}^2$. So, the projective GIT quotient for this action is

$$\varphi : (\mathbb{CP}^2)^{ss} \cong \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^2 /\!\!/ \mathbb{C}^* = \mathbb{CP}^1.$$

The bijection $\mathbb{CP}^2_{x \neq 0} \to \mathbb{C}^2$ given by $[x : y : z] \mapsto (y/x, z/x)$ implies that the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ is simply the usual scalar multiplication action. So, the $\mathbb{C}^*$-orbits in $(\mathbb{CP}^2)^{ss} = \mathbb{C}^2 \setminus \{0\}$ are punctured (at the origin) lines through the origin, which

[7] On the affine patch $\mathbb{CP}^2_{z \neq 0}$, this action is simply the action on $\mathbb{C}^2$ from Example 4.1.6.

[8] For the interested reader, this is a G-linearisation of the action. Another such G-linearisation is $\lambda \cdot (x, y, z) = (\lambda^{-2}x, y, z)$.

are closed in $\mathbb{C}^2 \setminus \{0\}$. Hence, the polystable set $(\mathbb{CP}^2)^{ps} = (\mathbb{CP}^2)^{ss}$. Furthermore, the stabiliser of a point $(a, b) \in \mathbb{C}^2 \setminus \{0\}$ is given by

$$\mathbb{C}^*_{(a,b)} = \{\lambda \in \mathbb{C}^* \mid \lambda \cdot (a, b) = (a, b)\} = \{1\}$$

which is finite, and so the stable set is equal to the polystable set which is equal to the semistable set:

$$(\mathbb{CP}^2)^s = (\mathbb{CP}^2)^{ps} = (\mathbb{CP}^2)^{ss} \cong \mathbb{C}^2 \setminus \{0\}.$$

Therefore, by Theorem 4.2.4 above, the projective GIT quotient is an orbit space quotient. This concludes the first part of the example.

We now illustrate the gluing process described earlier for this particular action. The semistable set for this action is given by

$$(\mathbb{CP}^2)^{ss} = (\mathbb{CP}^2)_{xy} \cup (\mathbb{CP}^2)_{xz} \cong U_0 \cup U_1 = \mathbb{C}^2 \setminus \{0\},$$

where $U_0 = \{(a, b) \in \mathbb{C}^2 \setminus \{0\} \mid a \neq 0\}$ and $U_1 = \{(a, b) \in \mathbb{C}^2 \setminus \{0\} \mid b \neq 0\}$ are affine patches covering $\mathbb{C}^2 \setminus \{0\}$. Without going through all the details, the coordinate algebra of U_0 is $\mathbb{C}[a^{\pm 1}, b]$ essentially because a being nonzero allows us to divide by a, so we include a^{-1} in the coordinate algebra. Similarly, the coordinate algebra of U_1 is $\mathbb{C}[a, b^{\pm 1}]$. In the intersection $U_0 \cap U_1 = \{(a, b) \in \mathbb{C}^2 \setminus \{0\} \mid a \neq 0 \text{ and } b \neq 0\}$, the coordinate algebra is $\mathbb{C}[a^{\pm 1}, b^{\pm 1}]$ since we can divide by both a and b.

Next, U_0 and U_1 are both clearly $\mathbb{C}^*$-invariant and so we can find their affine GIT quotients. Calculating the ring of invariants for each of the coordinate algebras is straightforward. For instance, the monomial $a^{-1}b = b/a$ in $\mathbb{C}[a^{\pm 1}, b]$ is clearly $\mathbb{C}^*$-invariant and so is contained in $\mathbb{C}[a^{\pm 1}, b]^{\mathbb{C}^*}$. In fact, it is fairly straightforward to show that all of the $\mathbb{C}^*$-invariant polynomials in $\mathbb{C}[a^{\pm 1}, b]$ are built from powers and linear combinations of $a^{-1}b$. Thus, we obtain $\mathbb{C}[b/a]$, $\mathbb{C}[a/b]$ and $\mathbb{C}[a/b, b/a] = \mathbb{C}[(a/b)^{\pm 1}]$ as the invariant rings of $\mathbb{C}[a^{\pm 1}, b]$, $\mathbb{C}[a, b^{\pm 1}]$ and $\mathbb{C}[a^{\pm 1}, b^{\pm 1}]$ respectively.

Finally, it is clear that the affine varieties corresponding to these rings are $\mathbb{C}^1_{b/a}$, $\mathbb{C}^1_{a/b}$, and $\mathbb{C}^1 \setminus \{0\}$. The diagram[9] below illustrates this process, where the double arrows denote application of the relevant functor to each object of the smaller

[9]This diagram was inspired by one of Daniel Chan's 'Adventures in Pure Mathematics' on his Youtube channel 'DanielChanMaths'.

diagram:

$$\begin{tikzcd}
U_0 & & U_1 \arrow[rr, Rightarrow, "\mathbb{C}[-]", "\text{AG Duality}"'] & & \mathbb{C}[a^{\pm 1}, b] \arrow[dr] & & \mathbb{C}[a, b^{\pm 1}] \arrow[dl] \\
& U_0 \cap U_1 \arrow[ul, hook] \arrow[ur, hook] \arrow[dd, Rightarrow, "{(-) /\!\!/ G}"', "\text{Affine GIT quotient}"'] & & & & \mathbb{C}[a^{\pm 1}, b^{\pm 1}] \arrow[dd, Rightarrow, "(-)^G", "\text{Ring of Invariants}"] & \\
\mathbb{C}^1_{b/a} & & \mathbb{C}^1_{a/b} & & \mathbb{C}[\frac{b}{a}] \arrow[dr] \arrow[ll, Rightarrow, "\mathrm{Spec}(-)"', "\text{AG Duality}"] & & \mathbb{C}[\frac{a}{b}] \arrow[dl] \\
& \mathbb{C}^1 \setminus \{0\} \arrow[ul] \arrow[ur] & & & & \mathbb{C}[\frac{a}{b}, \frac{b}{a}] &
\end{tikzcd}$$

Note that in the above diagram, 'AG Duality' is shorthand for the algebra-geometry duality. Moreover, the reversal of the direction of the arrows in the mini-diagrams which indicate the duality. Now, considering the bottom left mini-diagram, we identify $b/a \in \mathbb{C}^1_{b/a} \setminus \{0\}$ with $a/b \in \mathbb{C}^1_{a/b} \setminus \{0\}$ and then form the disjoint union

$$\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\}) /\!\!/ \mathbb{C}^* = \mathbb{C}^1_{b/a} \dot{\cup}\, \mathbb{C}^1_{a/b}.$$

The reader may recall that this final gluing process can be visualised as by gluing two complex planes together to form the Riemann sphere, as seen in Figure 3.1.

This example can be generalised relatively easily to $X = \mathbb{CP}^n$ (as done in [Hos12]), but the pictures we get for the general case are not nearly as pretty!

We conclude this section with some useful criteria for semistable, polystable and stable points that will be useful later. The reason they are useful is that the usual (algebraic) definition requires one to find the ring of invariants first, and this can sometimes be very difficult.

Theorem 4.2.6 (Topological Criterion for Semistability [MFK94])**.** *Let $X \subseteq \mathbb{CP}^n$ be a projective G-variety, where G is a reductive group. For $x \in X$, let $\tilde{x}$ be a point lying above x in the affine cone $\widetilde{X} \subseteq \mathbb{C}^{n+1}$. Then x is:*

1. Semistable *if and only if* $0 \notin \overline{G \cdot \tilde{x}}$*;*

2. Polystable *if and only if* $G \cdot \tilde{x}$ *is closed in* $\mathbb{C}^{n+1}$*;*

3. Stable *if and only if* $G \cdot \tilde{x}$ *is closed in* $\mathbb{C}^{n+1}$ *and the stabiliser* $G_{\tilde{x}}$ *is finite.*

The topological criterion allows us to work with G-orbits and stabilisers entirely within the affine cone, which can in many cases be more straightforward, especially if the ring of invariants is too challenging to compute.

In the following example, we use the topological criterion to find the semistable, polystable and stable subsets for Example 4.2.5. Note that we do so without reference to the ring of invariants. We have no doubt that this example is known and in print elsewhere, but as we did not see it, we have worked out the details here.

Example 4.2.7. Consider the action of $G = \mathbb{C}^*$ on $X = \mathbb{CP}^2$ as given in Example 4.2.5. The action of $\mathbb{C}^*$ on the affine cone $\mathbb{C}^3$ is given by

$$\lambda \cdot (x, y, z) = (\lambda^{-1}x, \lambda y, \lambda z).$$

Finding the $\mathbb{C}^*$-orbits is similar to Example 4.1.2, and, through a relatively straightforward case exhaustion, we arrive at the following list of orbits for this action[10]:

- $S_\alpha = \{(x, y, z) \in \mathbb{C}^3 \mid x^2yz = \alpha\}$, $\alpha \in \mathbb{C} \setminus \{0\}$, (closed);
- $S_\beta = \{(x, y, 0) \in \mathbb{C}^3 \mid \quad xy = \beta\}$, $\beta \in \mathbb{C} \setminus \{0\}$, (closed);
- $S_\gamma = \{(x, 0, z) \in \mathbb{C}^3 \mid \quad xz = \gamma\}$, $\gamma \in \mathbb{C} \setminus \{0\}$, (closed);
- Punctured (at the origin) x, y and z axes, (closure contains zero);
- Punctured (at the origin) lines in the yz-plane, (closure contains zero);
- Origin $\{(0, 0, 0)\}$, (closed).

Now consider a non-zero point $a = (x, y, z) \in \mathbb{C}^3 \setminus \{0\}$ in the affine cone. The possibilities for the $\mathbb{C}^*$-orbit of this point include any of the orbits in the above list except the origin. We can immediately exclude the punctured orbits since their closures contain the origin, and so by the topological criterion, these orbits correspond to the unstable points in $\mathbb{CP}^2$, which are given by

$$(\mathbb{CP}^2)^{us} = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0 \text{ or } y = z = 0\} = V(xy, xz).$$

Also, the closed orbits S_α, S_β and S_γ do not contain the origin, so they correspond to polystable points in $\mathbb{CP}^2$ by the topological criterion. The stabiliser of a is given by

$$\mathbb{C}^*_a = \{\lambda \in \mathbb{C}^* \mid \lambda \cdot (x, y, z) = (\lambda^{-1}x, \lambda y, \lambda z) = (x, y, z)\}$$

[10] One should visualise these orbits as if the field was $\mathbb{R}$ instead of $\mathbb{C}$, so in $\mathbb{R}^3$.

and at least one of x, y or z must be non-zero, so $\mathbb{C}^*_a$ is necessarily trivial. Thus, according to the topological criterion again, the stable, polystable and semistable sets all coincide. Hence, we obtain

$$(\mathbb{CP}^2)^s = \mathbb{CP}^2 \setminus (\mathbb{CP}^2)^{us} = \{[x : y : z] \in \mathbb{CP}^2 \mid x \neq 0 \text{ and } (y, z) \neq 0\}.$$

Of course, the topological criterion was not necessary in the above example, since the ring of invariants was relatively easy to find in Example 4.2.5 anyway.

The next important criterion for semistability and stability is called the Hilbert-Mumford criterion. Let $X \subseteq \mathbb{CP}^n$ be a projective G-variety, where G is a reductive group. The main idea is that in order to find semistable, polystable or stable points in X, it suffices to compute these stability conditions for one-parameter subgroups (1-PS) of G. That is, for non-trivial (affine algebraic) group homomorphisms[11]

$$\lambda : \mathbb{C}^* \to G.$$

Theorem 4.2.8 (Hilbert-Mumford Criterion [MFK94])**.** *For a projective G-variety $X \subseteq \mathbb{CP}^n$, where G is a reductive group, a point $x \in X$ is (semi)stable if and only if x is (semi)stable for all one-parameter subgroups $\lambda : \mathbb{C}^* \to G$.*

Given such a 1-PS λ, one can study its asymptotic behaviour on a point $\tilde{x}$ lying above $x \in X$ in the affine cone $\widetilde{X} \subseteq \mathbb{C}^{n+1}$, as the parameter $t \in \mathbb{C}^*$ tends to zero and infinity. Doing so to deduce whether x is (semi)stable with respect to λ. The Hilbert-Mumford criterion thus yields another interpretation of reductivity. Namely, reductive groups contain enough 1-PS to completely detect stability of points.[12] We recommend to the reader the notes by Thomas in [Tho06, p. 11] for an illuminating picture explaining the Hilbert-Mumford criterion, and Hoskins [Hos12, p. 30] for a more detailed exposition.

We have now reached the end of Part II of our story. To recap, in Chapter 3 we scoured the shallows of the algebraic geometry sea. Following that, in Chapter 4 we probed, with examples, the intricacies of the theory of quotients in algebraic geometry known as Geometric Invariant Theory.

Marching onward to the next piece of our story, we delve into the intimate connection between the quotients within the worlds of symplectic geometry and algebraic geometry.

[11] Group homomorphisms that are affine morphisms.

[12] This is in fact true over arbitrary fields, as long as one uses the appropriate definition of reductive there.

Part III

Linking Symplectic and Algebraic Quotients

Chapter 5

The Guillemin-Sternberg Theorem

"Algebraic geometry is a more rigid world, whereas symplectic geometry is more flexible. That's one reason they're such different worlds, and it's so surprising they end up being equivalent in a deep sense."

– Nick Sheridan, *Mathematicians Explore Mirror Link Between Two Geometric Worlds*, 2018 [Har18].

At long last, we arrive at the final chapter of our story. Here, we illuminate the link between the quotients of symplectic geometry and algebraic geometry first discovered by Mumford, Guillemin, and Sternberg in the paper *Geometric quantization and multiplicities of group representations*, authored by Guillemin and Sternberg in 1982 [GS82].

In this chapter, we showcase Guillemin and Sternberg's discovery, which here we call the *Guillemin-Sternberg Theorem*, as stated in the introduction. We will mostly follow the arguments given in Chapter 8 of Mumford's book *Geometric Invariant Theory* [MFK94]. There, the work of Kempf and Ness (from their 1979 paper *The length of vectors in representations spaces* [KN79]) is employed extensively. As such, we also utilise Kempf and Ness' work as we endeavour towards the discovery of Guillemin and Sternberg.

The primary goal of this chapter is to prove the Guillemin-Sternberg Theorem[1]:

Theorem. (Guillemin and Sternberg)**.** *Suppose that $X \subseteq \mathbb{CP}^n$ is a smooth projective G-variety, where G is the complexification of a compact connected real Lie group K. Denote the Fubini-Study moment map corresponding to the Hamiltonian K-manifold X by $\mu : X \to \mathfrak{K}^*$.*

With this setup, we have $\mu^{-1}(0) \subseteq G \cdot \mu^{-1}(0) = X^{ps}$ and this inclusion induces a homeomorphism

$$\mu^{-1}(0)/K \to X /\!\!/ G$$

between the symplectic quotient and the GIT quotient.

[1] The (perhaps unknown) terminology and concepts presented in the Guillemin-Sternberg Theorem will be explained within this chapter.

We importantly remark that we have attempted to isolate the fundamental constituents behind the Guillemin-Sternberg Theorem as best we could. In particular, we found that a great deal of headway can be made without invoking GIT.

Before we move onto the general setup described at the start of the Guillemin-Sternberg Theorem above, we first give a motivating example that illustrates much of what we shall later see.

Example 5.0.1. Consider the action of the compact connected Lie group $K = S^1$ on $X = \mathbb{CP}^2$ induced by the action of K on $\mathbb{C}^3$ given as

$$\lambda \cdot (x, y, z) = (\lambda^{-1}x, \lambda y, \lambda z).$$

That is, the action of K on $\mathbb{CP}^2$ is

$$\lambda \cdot [x : y : z] = [\lambda^{-1}x : \lambda y : \lambda z].$$

For this action, note that K may be considered a subgroup of $\mathrm{U}(3)$ via the unitary representation $\lambda \mapsto \mathrm{diag}(\lambda^{-1}, \lambda, \lambda) \in \mathrm{U}(3)$. Moreover, the corresponding Lie algebra representation $\mathrm{Lie}(S^1) \to \mathfrak{u}(3)$ is given by $A \mapsto \mathrm{diag}(-A, A, A)$ for $A \in \mathrm{Lie}(S^1) \cong i\mathbb{R}$. According to Theorem 2.4.9, this action of K on $\mathbb{CP}^2$ is Hamiltonian with moment map $\mu : \mathbb{CP}^2 \to \mathrm{Lie}(S^1)^*$ given by

$$\mu^A([x : y : z]) = \frac{h((-Ax, Ay, Az), (x, y, z))}{2ih((x, y, z), (x, y, z))} = \frac{A(-|x|^2 + |y|^2 + |z|^2)}{2i(|x|^2 + |y|^2 + |z|^2)}.$$

Then, the zero level set of μ is given by

$$\mu^{-1}(0) = \{[x : y : z] \in \mathbb{CP}^2 \mid |y|^2 + |z|^2 = |x|^2\}.$$

Note that we must have $x \neq 0$ in $\mu^{-1}(0)$, for otherwise, this would imply that $x = y = z = 0$ which is impossible in $\mathbb{CP}^2$. Thus, we have

$$\mu^{-1}(0) = \left\{ \left[1 : \frac{y}{x} : \frac{z}{x}\right] \in \mathbb{CP}^2_{x \neq 0} \,\middle|\, \left|\frac{y}{x}\right|^2 + \left|\frac{z}{x}\right|^2 = 1 \right\} \cong S^3 \subseteq \mathbb{C}^2.$$

Hence, the symplectic reduction of $\mathbb{CP}^2$ by S^1 (for this action) is

$$\mu^{-1}(0)/K \cong S^3/S^1 \cong \mathbb{CP}^1.$$

Next, observe that the action above extends an action of $G = \mathbb{C}^*$ (we will soon see, in Definition 5.0.4, that G is the called the *complexification* of K) on

$X = \mathbb{CP}^2$ via the linear representation $\rho : \mathbb{C}^* \to \mathrm{GL}_3(\mathbb{C})$, as in Example 4.2.5 from the Chapter 4. In that example, we saw that the semistable set, polystable set, and stable set all coincided and were equal to $\mathbb{C}^2 \setminus \{0\}$, which contains the zero level set $\mu^{-1}(0) = S^3$.

Now we make the keen observation that

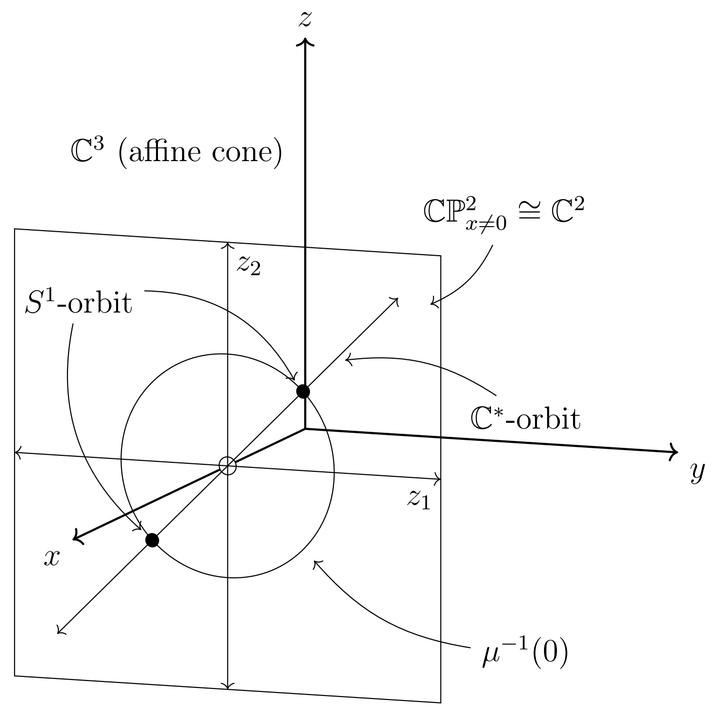
$$G \cdot \mu^{-1}(0) = \mathbb{C}^* \cdot S^3 = \mathbb{C}^2 \setminus \{0\} = X^{ps},$$

where X^{ps} is the polystable set. To see why this is true, note that G dilates the unit 3-sphere to give all the 3-spheres of varying radii, whose union is the punctured affine plane $\mathbb{C}^2 \setminus \{0\}$.

We further observe that each G-orbit of a point, contained in the polystable set $\mathbb{C}^2 \setminus \{0\}$, intersects the zero level set $\mu^{-1}(0) = S^3$ in a unique K-orbit. To see this, we argue as follows. For $(a, b) \in \mathbb{C}^2 \setminus \{0\}$, we have

$$\begin{aligned}
\mathbb{C}^* \cdot (a,b) \cap S^3 &= \{(\lambda a, \lambda b) \in \mathbb{C}^2 \setminus \{0\} \mid \lambda \in \mathbb{C}^* \text{ and } |\lambda a|^2 + |\lambda b|^2 = 1\} \\
&= \{(\lambda a, \lambda b) \in \mathbb{C}^2 \setminus \{0\} \mid \lambda \in \mathbb{C}^* \text{ and } \left|\lambda(|a|^2 + |b|^2)^{\frac{1}{2}}\right| = 1\} \\
&= \{(sa_0, sb_0) \in \mathbb{C}^2 \setminus \{0\} \mid |s| = 1\} \\
&= S^1 \cdot (a_0, b_0),
\end{aligned}$$

where $a_0 = a/(|a|^2 + |b|^2)^{1/2}$ and $b_0 = b/(|a|^2 + |b|^2)^{1/2}$. The following diagram is a picture of the real locus describing the above situation:



Finally, we saw that in Example 4.2.5 the (projective) GIT quotient of $\mathbb{CP}^2$ by $\mathbb{C}^*$ (for this action) is $\mathbb{CP}^1$. Thus, we observe that the GIT quotient of X by G and the symplectic reduction of X by K are identical:

$$X /\!\!/ G = \mathbb{CP}^2 /\!\!/ \mathbb{C}^* \cong \mathbb{CP}^1 \cong S^3/S^1 \cong \mu^{-1}(0)/K.$$

The above observations illustrated in this introductory example were not coincidental. The remainder of this chapter is dedicated to proving the highlighted observations in the general setting described at the start of this chapter in the Guillemin-Sternberg Theorem.

To remind the reader of the setting, we consider an arbitrary *smooth* complex projective variety $X \subseteq \mathbb{CP}^n$ that is acted upon by both a compact connected real Lie group K, and by its complexification G. The complexification G is, as we shall see, a reductive group. In this setting, we first show that X is naturally a symplectic manifold, and with the use of a number of intermediate results, we will prove that the symplectic reduction of X by K is homeomorphic to the projective GIT quotient of X by G (in other words, the conclusion of the Guillemin-Sternberg Theorem).

Without further ado, let $X \subseteq \mathbb{CP}^n$ be a complex projective variety. Recall from Example 2.2.17 in Chapter 2, that $\mathbb{CP}^n$ is a smooth manifold. If the polynomial equations defining X are such that their Jacobian has a non-vanishing determinant, then by the Regular Level Set Theorem, it follows that X is a smooth manifold. For the rest of this chapter we consider *smooth* complex projective varieties in $\mathbb{CP}^n$. The first important consequence of this smoothness assumption, is the following proposition.

Proposition 5.0.2. *A smooth complex projective variety $X \subseteq \mathbb{CP}^n$ is a symplectic manifold with symplectic form given by restriction of the Fubini-Study form ω_{FS} on $\mathbb{CP}^n$.*

Proof. Let $\iota : X \hookrightarrow \mathbb{CP}^n$ be the inclusion map. We need to show that $\omega = \iota^*\omega_{FS}$ is a symplectic form on X. Recall that the pullback of a 2-form is also a 2-form, and since the exterior derivative commutes with pullbacks, it follows that ω is closed. To prove the non-degeneracy of ω, take $x \in X$ and let $v \in T_xX$ be non-zero. Note that $iv \in T_xX$ since T_xX is a complex vector space. We then have $\omega_x(v, iv) = (\omega_{FS})_x(v, iv)$, which is strictly positive by a property of the Fubini-Study form.[2] $\square$

Now consider a compact connected Lie subgroup K of $\mathrm{U}(n+1)$ which acts on $\mathbb{C}^{n+1}$ by restriction of the standard representation. We remind the reader that, from Theorem 2.4.9, the action of K on $\mathbb{C}^{n+1}$ induces a Hamiltonian action of K on the symplectic manifold $(\mathbb{CP}^n, \omega_{FS})$ with Fubini-Study moment map $\mu : \mathbb{CP}^n \to \mathfrak{K}^*$ given by

$$\mu^A([z]) = \frac{h(Az, z)}{2ih(z, z)},$$

for $A \in \mathfrak{K}^*$ and $z \in \mathbb{C}^{n+1} \setminus \{0\}$. Where in the above, h denotes the standard Hermitian inner product on $\mathbb{C}^{n+1}$ and ω_{FS} denotes the Fubini-Study symplectic form on $\mathbb{CP}^n$. In other words, $\mathbb{CP}^n$ is a Hamiltonian K-manifold. In general, it turns out that any smooth complex projective variety $X \subseteq \mathbb{CP}^n$ is also a Hamiltonian K-manifold, as we now show.

[2]We have omitted proof or statement about this property elsewhere in this thesis since it involves the use of Kähler manifolds. It turns out that $(\omega_{FS})_x(v, iv)$ comes from a Riemannian metric that is compatible with the symplectic form, and is thus positive definite on tangent spaces.

Proposition 5.0.3. *For a smooth complex projective variety $X \subseteq \mathbb{CP}^n$, the restriction of the action of K on $\mathbb{CP}^n$ to X is Hamiltonian with moment map* (which we also call the Fubini-Study moment map) *given by the composition*

$$X \xhookrightarrow{\iota} \mathbb{CP}^n \xrightarrow{\mu} \mathfrak{K}^*$$

where ι denotes the inclusion map of X into $\mathbb{CP}^n$.

Proof. Denote the symplectic form on X induced by the Fubini-Study form on $\mathbb{CP}^n$ as $\omega = \iota^*\omega_{FS}$. The action of K on X is symplectic by the properties of pullbacks and the fact that the action of K on $\mathbb{CP}^n$ is symplectic.

The K-equivariance of $\mu \circ \iota$ follows from the K-equivariance of μ:

$$(\mu \circ \iota)(k \cdot x) = \mu(k \cdot x) = \mathrm{Ad}_k^* \mu(x) = \mathrm{Ad}_k^* (\mu \circ \iota)(x)$$

for $k \in K$ and $x \in X$. Let $A \in \mathfrak{K}^*$ and note that $(\mu \circ \iota)^A = \mu^A \circ \iota = \iota^*\mu^A$. Now via the properties of pullbacks and the definition of a moment map, we obtain

$$d(\mu \circ \iota)^A = d(\iota^*\mu^A) = \iota^* d\mu^A = \iota^*(i_{A_{\mathbb{CP}^n}}\omega_{FS}) = i_{A_X}(\iota^*\omega_{FS}) = i_{A_X}\omega$$

and so $\mu \circ \iota$ satisfies Hamilton's equation. Thus, $\mu \circ \iota$ is a moment map for the action of K on X. $\square$

Now that we have established the fact that any smooth complex projective variety $X \subseteq \mathbb{CP}^n$ is a Hamiltonian K-manifold, we can continue building the bridge to the world of algebraic geometry. Suppose K is a compact connected Lie subgroup of the unitary group $\mathrm{U}(n+1)$. We now define the *complexification* of K.

Definition 5.0.4 (Lie group complexification). *The* complexification *of K is a complex Lie group $G \subseteq \mathrm{GL}_{n+1}(\mathbb{C})$ containing K, with the property that the Lie algebra $\mathfrak{g}$ of G is the complexification*[3] *of the Lie algebra $\mathfrak{K}$ of K* (i.e., $\mathfrak{g} \cong \mathfrak{K} \oplus i\mathfrak{K}$). *We denote the complexification of K by either G or $K_{\mathbb{C}}$.*

Some examples of compact Lie groups K and their complexifications G include the pairs (G, K):

$$(\mathbb{C}^*, S^1),\ (\mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n)),\ (\mathrm{SL}_n(\mathbb{C}), \mathrm{SU}(n)).$$

The complexification of compact connected Lie groups offers us two important, and useful properties, which we employ soon.

[3]By complexification of a Lie algebra, we mean the complexification of its underlying vector space structure.

Theorem 5.0.5. *Let $G = K_{\mathbb{C}}$ be the complexification of a compact connected Lie subgroup K of* $\mathrm{U}(n+1)$*. Then,*

1. *G is a complex reductive group;*
2. (Cartan decomposition)*: For any $g \in G$, there exists $k \in K$ and $A \in \mathfrak{K}$ such that*
$$g = k \exp(iA).$$

We will not be proving this theorem here due to space constraints. A proof of the first part of the above theorem can be found in *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras* by Gorbatsevich [Gor94, Chapter 4 Section 2.5]. For a proof of the second part of the above theorem (Cartan decomposition), the curious reader can seek out the book *Differential geometry, Lie groups, and symmetric spaces* by Helgason [Hel78, Chapter VI Theorem 1.1].

As the reader may suspect, the first property in the above theorem (that being a complexified compact connected Lie group is reductive) is fundamental in the bridge between the symplectic and algebra-geometric worlds that we are exploring in this thesis.

To re-orient ourselves, on the one hand, we have a compact connected Lie group $K \subseteq \mathrm{U}(n+1))$ acting on $\mathbb{C}^{n+1}$ via restriction of the standard representation. In light of Proposition 5.0.3 above, such an action induces a Hamiltonian action of K on a smooth complex projective variety $X \subseteq \mathbb{CP}^n$. Thus, we may consider the symplectic quotient (symplectic reduction) of X by K. On the other hand, by the above theorem, the complexification $G = K_{\mathbb{C}}$ is reductive and so X is a projective G-variety. Thus, we may consider its GIT quotient $X /\!\!/ G$.

One of the essential keys to the connection between the symplectic quotient and the GIT quotient lies in the following map:

$$p_z : G \to \mathbb{R}, \;\; p_z(g) = \|g \cdot z\|^2$$

where $\|\cdot\|$ denotes the norm on $\mathbb{C}^{n+1}$ induced by h and $z \in \mathbb{C}^{n+1}$. These maps were first considered by Kempf and Ness [KN79] in order to understand how lengths change along the G-orbits. We will call these maps *Kempf-Ness* functions, and we now outline some of their properties.

We say that $g \in G$ is a *critical point* of the function p_z if its derivative $d_g p_z : T_g G \to T_{\|g \cdot z\|^2}\mathbb{R}$ of p_z at g vanishes. Also, we say that p_z is *convex* if its restriction to the one-parameter subgroups of G are each convex functions of a single real variable. The first property we outline, of a Kempf-Ness function, is as follows.

Proposition 5.0.6. *For any $z \in \mathbb{C}^{n+1} \setminus \{0\}$, the Kempf-Ness function p_z is convex (so every critical point is a minimum). Furthermore, we have that $g \in G$ is a critical point of p_z if and only if $\mu(g \cdot [z]) = 0$.*

Proof. The following proof is adapted from the proof given by Hoskins in [Hos12] with some omitted details now included.

Suppose $z \in \mathbb{C}^{n+1} \setminus \{0\}$. First we note that p_z is constant on K since h is K-invariant. In particular, this implies that $p_z(\exp(Z)) = p_z(\exp(iA))$ for any $Z = A' + iA \in \mathfrak{K} \oplus i\mathfrak{K} \subseteq \mathfrak{g}$, since G is the complexification of K.

Thus, to show that p_z is convex, we just need to show that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} p_z(\exp(itA)) \geqslant 0$$

for every $A \in \mathfrak{K}$. We have

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} p_z(\exp(itA)) &= \left.\frac{d}{dt}\right|_{t=0} [h(iA\exp(itA)z, \exp(itA)z) \\ &\qquad + h(\exp(itA)z, iA\exp(itA)z)] \\ &= h((iA)^2 z, z) + h(iAz, iAz) + h(iAz, iAz) + h(z, (iA)^2 z) \\ &= 4h(iAz, iAz) = 4\left\|Az\right\|^2 \geqslant 0\end{aligned}$$

where in the last line we have used the fact that iA is self-adjoint, and so p_z is convex.

For the second claim, the equality $p_z(g) = p_{g\cdot z}(e)$ implies that $d_g p_z = d_e p_{g\cdot z}$ and so it suffices to prove that e is a critical point of p_z if and only if $\mu([z]) = 0$. By doing this we can use the Lie algebra of G to make the computation easier. As a map, we have $d_e p_z : T_e G\,(\cong \mathfrak{g}) \to T_{\|z\|^2}\mathbb{R}\,(\cong \mathbb{R})$, so as we established in the proof of the first claim, we only need to compute the derivative on $i\mathfrak{K} \subseteq \mathfrak{g}$. For $A \in \mathfrak{K}$, we calculate

$$\begin{aligned}d_e p_z(iA) &= \left.\frac{d}{dt}\right|_{t=0} p_z(\exp(itA)) = \left.\frac{d}{dt}\right|_{t=0} h(\exp(itA)z, \exp(itA)z) \\ &= h\left(\left.\frac{d}{dt}\right|_{t=0} \exp(itA)z, z\right) + h\left(z, \left.\frac{d}{dt}\right|_{t=0} \exp(itA)z\right) \\ &= h(iAz, z) + h(z, iAz) \\ &= 2ih(Az, z)\end{aligned}$$

where in the last line we used the fact that iA is self-adjoint. Thus, we obtain

$$d_e p_z(iA) = 2ih(Az, z) = -4h(z, z)\mu^A([z]),$$

and the conclusion follows. $\square$

The property described in the above proposition of Kempf-Ness functions in particular allows us to prove the following proposition.

Proposition 5.0.7. *If a G-orbit intersects the zero level set $\mu^{-1}(0)$, then they intersect in a unique K-orbit.*

Proof. Suppose that $x, y \in X$ both lie in the same G-orbit and in the zero level set $\mu^{-1}(0)$. In other words, there exists $g \in G$ such that $x = g \cdot y$ and $\mu(x) = 0 = \mu(y)$. To show that they lie in the same K-orbit, the idea is to utilise the convexity of the Kempf-Ness functions from the previous proposition. Let $\tilde{x}$ and $\tilde{y}$ denote the points lying above x and y respectively in the affine cone $\widetilde{X} \subseteq \mathbb{C}^{n+1}$; that is, $[\tilde{x}] = x$ and $[\tilde{y}] = y$.

It follows from the previous proposition that $e \in G$ minimises $p_{\tilde{y}}$ and $g \in G$ also minimises $p_{\tilde{y}}$ since $\mu(g \cdot [\tilde{y}]) = \mu([\tilde{x}]) = 0$. Hence, we obtain $p_{\tilde{y}}(g) = p_{\tilde{y}}(e)$.

By virtue of the Cartan decomposition theorem (Theorem 5.0.5), there exists $k \in K$ and $A \in \mathfrak{K}$ such that $g = k \exp(iA)$ and as h is K-invariant, we have

$$p_{\tilde{y}}(g) = \|g \cdot \tilde{y}\|^2 = \|\exp(iA) \cdot \tilde{y}\|^2 = p_{\tilde{y}}(\exp(iA)).$$

Now, the convexity of $p_{\tilde{y}}$ ensures that

$$p_{\tilde{y}}(\exp(itA)) \leqslant t p_{\tilde{y}}(\exp(iA)) + (1-t) p_{\tilde{y}}(\exp(0)) = p_{\tilde{y}}(e)$$

for every $t \in [0, 1]$; but as e minimises $p_{\tilde{y}}$, this inequality must be an equality. This implies that

$$0 = \left.\frac{d}{dt}\right|_{t=0} p_{\tilde{y}}(e) = \left.\frac{d}{dt}\right|_{t=0} p_{\tilde{y}}(\exp(itA)) = 2\left\|A\tilde{y}\right\|^2$$

and so the infinitesimal action of A on y, $(A_X)_y = (A_{\mathbb{C}^{n+1}})_{\tilde{y}} = A\tilde{y}$, vanishes. In other words, $iA \in i\mathfrak{K}_y \subseteq \mathfrak{g}_y = \mathrm{Lie}(G_y)$ so $\exp(iA) \in G_y$. Thus, we conclude that $x = g \cdot y = k \exp(iA) \cdot y = k \cdot y$, and therefore x and y are in the same K-orbit. $\square$

Significantly, we note that we have not used GIT so far for any of the proofs in this section. We will soon see that the G-orbits that do intersect the zero level set of the moment map are precisely the polystable ones. Beforehand, we prove that the minima of Kempf-Ness functions only exist on closed orbits, and that the closure of orbits is sufficient to guarantee their existence. We remark that this result was initially proved by Kempf and Ness in [KN79] and does in fact invoke GIT.

Recall that a map $f : X \to Y$ between topological spaces is proper if the preimage of every compact set is compact.

Proposition 5.0.8. *For $z \in \mathbb{C}^{n+1} \setminus \{0\}$, the Kempf-Ness function p_z attains a minimum if and only if the G-orbit $G \cdot z$ is Zariski-closed.*

Proof. For the following proof, we will only prove the forward implication formally, and sketch the main ideas for the converse. The full details can be found in Hoskins' notes [Hos12, Lemma 10.8 and Lemma 10.10], but some minor tweaks and corrections have been made here.

Fix $z \in \mathbb{C}^{n+1} \setminus \{0\}$ and suppose that $G \cdot z$ is Zariski-closed and hence closed in the Euclidean topology. We first note that the map $\|\cdot\|^2 : \mathbb{C}^n \to \mathbb{R}$ is proper since the preimage of every bounded set is bounded and the preimage of every closed set is closed by continuity. Thus, $\|\cdot\|^2$ is closed by Proposition 2.2.11; so if $G \cdot z$ is closed then $\|G \cdot z\|^2$ is closed in $\mathbb{R}$. Therefore,

$$\inf_{g \in G} p_z(g) \in \overline{\|G \cdot z\|^2} = \|G \cdot z\|^2 = p_z(G)$$

and so p_z attains a minimum.

For the converse, which now invokes GIT, suppose that $G \cdot z$ is not Zariski-closed. We need to show that p_z does not attain a minimum, so it suffices to show that p_z has no critical points. Then by a standard fact about algebraic groups, one can find a closed orbit $G \cdot w \subseteq \overline{G \cdot z} \setminus G \cdot z$ (see Chevalley's theorem for constructible sets in Hartshorne [Har77, Chapter II Exercise 3.19]). Next, with the Hilbert-Mumford criterion one can show that there is a 1-PS $\lambda : \mathbb{C}^* \to G$ such that, by conjugating λ with a suitable $g \in G$ (see Helgason [Hel78, Chapter VI Theorem 2.1]), we have $\lambda(S^1) \subseteq K$ and that

$$\lim_{t \to 0} \lambda(t) \cdot z \in G \cdot w,$$

where the limit is taken in the usual analytic sense over $\mathbb{C}$.[4] Then, using the fact that $\mathbb{C}^*$ is reductive, its action on $V = \mathbb{C}^{n+1}$ via λ is completely reducible, and so we have a decomposition of V into a direct sum of the form

$$V = \bigoplus_{r \in \mathbb{Z}} V_r,$$

where each V_r is a direct sum of an irreducible representation with a particular irreducible character. As $\mathbb{C}^*$ is abelian, it follows that these irreducible characters

[4] Strictly speaking, this has limit has a more formal definition over arbitrary fields. As we are working over $\mathbb{C}$ however, this is not necessary.

are the one-dimensional representations of $\mathbb{C}^*$, and so are of the form $t \mapsto t^r$ for some $r \in \mathbb{Z}$. Thus, we have

$$V_r = \{u \in V \mid \lambda(t) \cdot u = t^r u \text{ for all } t \in \mathbb{C}^*\},$$

and the decomposition is called a weight-space decomposition, where the integers $r \in \mathbb{Z}$ are called weights.

Furthermore, this weight-space decomposition is orthogonal with respect to h. That is, $h(v_r, v_s) = 0$ for $v_r \in V_r$ and $v_s \in V_s$ where r and s are distinct integers. To see why, for every $t \in S^1$ we have

$$h(v_r, v_s) = h(\lambda(t) \cdot v_r, \lambda(t) \cdot v_s) = h(t^r v_r, t^s v_s) = t^{r-s} h(v_r, v_s)$$

since $\lambda(S^1) \subseteq K$ and h is K-invariant. Thus, $h(v_r, v_s) = 0$ as $r \neq s$.

With respect to this decomposition, we have $z = \sum_{r \in \mathbb{Z}} z_r$ and so

$$\lambda(t) \cdot z = \sum_{r \in \mathbb{Z}} \lambda(t) \cdot z_r = \sum_{r \in \mathbb{Z}} t^r z_r.$$

Upon taking limits, and noting that the limit exists it follows that $z_r = 0$ for all $r < 0$. Furthermore, as the limit is not contained in $G \cdot z$, it must be the case that $z_r \neq 0$ for some $r > 0$. For if this was not the case, we would have $z = 0$, and then the limit would be zero, or the action would be trivial.

To show that p_z does not have a critical point, we need to show that $d_g p_z \neq 0$ for all $g \in G$, or equivalently, that $d_e p_{g \cdot z} \neq 0$ for all $g \in G$ since $p_{g \cdot z}(e) = p_z(g)$. By considering the following element

$$A := \left.\frac{d}{dt}\right|_{t=0} \lambda(e^{it})$$

of the Lie algebra of K, which belongs to $\mathfrak{K}$ as $\lambda(S^1) \subseteq K$, the infinitesimal action of A on z_r is given by

$$(A_{\mathbb{C}^n})_{z_r} = A z_r = \left.\frac{d}{dt}\right|_{t=0} \lambda(e^{it}) \cdot z_r = \left.\frac{d}{dt}\right|_{t=0} e^{irt} z_r = ir z_r.$$

Thus, as the decomposition of V is orthogonal with respect to h, we obtain

$$h(Az, z) = \sum_{r,s \in \mathbb{Z}} h(A z_r, z_s) = \sum_{r,s \in \mathbb{Z}} ir h(z_r, z_s) = \sum_{r=1}^{\infty} ir \left\| z_r \right\|^2$$

and so

$$d_e p_z(iA) = 2ih(Az, z) = -2 \sum_{r=1}^{\infty} r \left\| z_r \right\|^2 < 0$$

as at least one z_r is non-zero. Hence, e is not a critical point of p_z. Repeating the above argument, replacing z with $g \cdot z$, we have that e is not a critical point of $p_{g \cdot z}$. Thus, p_z has no critical points and so we are done. □

Before stating we prove the Guillemin-Sternberg Theorem, we remind the reader of the necessary setup. Let K be a compact connected Lie subgroup of $\mathrm{U}(n+1)$ and let $G < \mathrm{GL}_{n+1}(\mathbb{C})$ be the complexification of K, which is a complex reductive group. Suppose $X \subseteq \mathbb{CP}^n$ is a smooth complex projective G-variety. Then (X, ω) is also Hamiltonian K-manifold, where ω is the restriction of the Fubini-Study form on $\mathbb{CP}^n$. Let $\mu : X \to \mathfrak{K}^*$ denote corresponding moment map for this Hamiltonian action.

Theorem 5.0.9 (Guillemin and Sternberg)**.** *With the above setup, we have the set equality $G \cdot \mu^{-1}(0) = X^{ps}$ and the inclusion of $\mu^{-1}(0)$ in X^{ps} induces a homeomorphism*

$$\mu^{-1}(0)/K \to X /\!\!/ G$$

between the symplectic quotient and the GIT quotient.

Proof. The following proof is based on Hoskins [Hos12, Theorem 10.3]. For $x \in X$ and $g \in G$, let $\tilde{x}$ denote a point lying above x in the affine cone $\widetilde{X} \subseteq \mathbb{C}^{n+1}$. Then, we have

$$\begin{aligned} g \cdot x \in \mu^{-1}(0) &\iff p_{\tilde{x}} \text{ has a critical point at } g && \text{(Proposition 5.0.6)} \\ &\iff G \cdot \tilde{x} \text{ is closed} && \text{(Proposition 5.0.8)} \\ &\iff x \in X^{ps} \end{aligned}$$

where the last implication is due to the topological criterion for semistability (see Theorem 4.2.6). Thus, $x \in X^{ps}$ if and only if $x \in g^{-1}\mu^{-1}(0) \subseteq G \cdot \mu^{-1}(0)$, which completes the proof of the first claim.

For the second claim, we observe that the quotient map $\phi : X^{ps} \to X^{ps}/G$ is continuous and G-invariant. Thus, by the universal property of the quotient topology, the inclusion $\mu^{-1}(0) \subseteq X^{ps}$ induces a continuous map $\psi : \mu^{-1}(0)/K \to X^{ps}/G$ such that the following diagram commutes:

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook] \arrow[d, "\pi"'] & X^{ps} \arrow[d, "\phi"] \\ \mu^{-1}(0)/K \arrow[r, "\psi"] & X^{ps}/G. \end{tikzcd}$$

To see that ψ is surjective, let $y \in X^{ps}/G$ be arbitrary. As ϕ is surjective, there exists $x \in X^{ps}$ such that $\phi(x) = y$. From the first part of the claim, $x = g \cdot x'$ for some $x' \in \mu^{-1}(0)$ and so $\psi(\pi(x')) = \phi(x') = \phi(x)$ since ϕ is G-invariant; thus ψ is surjective. For injectivity of ψ, suppose that $\psi(\pi(x)) = \psi(\pi(x'))$ for some $x, x' \in \mu^{-1}(0)$. Then $\phi(x) = \phi(x')$ which implies that x and x' belong to the same G-orbit. From Proposition 5.0.7 it follows that x and x' belong to the same K-orbit and hence ψ is injective.

Note that X^{ps}/G is Hausdorff according to the homeomorphism $X^{ps}/G \cong X /\!\!/ G$ from Proposition 4.2.4. Moreover, $\mu^{-1}(0)/K$ is compact since $\mu^{-1}(0)$ is a closed subset of the compact space $X \subseteq \mathbb{CP}^n$ and so is also compact. Thus, ψ is a continuous bijection from a compact space to a Hausdorff space and hence is a homeomorphism. $\square$

So we have proved the Guillemin-Sternberg Theorem, concluding our story about the link between the symplectic and algebraic quotients. Namely, under suitable conditions, the symplectic reduction from symplectic geometry and the GIT quotient from algebraic geometry are homeomorphic. This appears to be a purely topological result, since the correspondence is given by a homeomorphism of the underlying topological spaces in each quotient.

We wonder whether the Guillemin-Sternberg Theorem can be rephrased, or extended, to a categorical statement linking some subcategory of the category of symplectic manifolds with some subcategory of the category of complex projective varieties.[5] We also wonder if there is an even deeper connection, such as an analogue of the Guillemin-Sternberg Theorem for projective varieties (or schemes) over arbitrary fields and some suitable analogue of symplectic manifolds.[6]

[5] The author is aware of the potential absurdity of such a näive question, but we ask it anyway.

[6] This näive question may also be an absurd one to ask.

Conclusion

The homeomorphism of the symplectic reduction from symplectic geometry, and the GIT quotient from algebraic geometry, as given by the Guillemin-Sternberg Theorem, provides a bridge between two seemingly disparate geometric worlds.

On the one hand, symplectic geometry has its roots in classical mechanics. While on the other hand, algebraic geometry has its roots in the study of solutions to polynomial equations in many variables. Each geometric theory has been traditionally described in quite different languages, fogging any clear pathway between them. So, from a näive and uneducated perspective, we were initially surprised that such a track exists and had been uncovered.

However, upon close examination of the road towards the Guillemin-Sternberg theorem, some key crucial landmarks were revealed to us. These include, for instance, the correspondence of complex reductive groups and compact Lie groups, the properties of the Kempf-Ness functions defined in terms of a Hermitian inner product on complex Euclidean space, and the fact that any smooth, complex projective variety can be equipped with a symplectic structure. From the perspective of these way-points, and the benefit of hindsight, such a connection is perhaps not as bizarre as we had initially believed.

In light of this revelation, perhaps a more natural setting for this result is Kähler geometry, which we, unfortunately, had to omit introducing in this thesis due to its already broad scope. The rationale for entertaining this thought is partly due to the Géometrie Algébrique et Géométrie Analytique principle of Serre [Ser56], which describes an intimate relationship between complex geometry and algebraic geometry. Furthermore, the rationale is also partly due to the fact the natural symplectic structure on a smooth, complex projective variety is derived from a Hermitian inner product on complex Euclidean space, which turns out to be a Kähler metric from Kähler geometry.

If it is the case that Kähler geometry is a more natural setting for the Guillemin-Sternberg theorem, we ask if the theorem has a natural analogue within the setting of Riemannian geometry in the form of a relationship between Riemannian manifold quotients and GIT quotients.

Among other topics we unfortunately had to omit include the language of schemes and sheaves within algebraic geometry, and particularly the notion of linearisations of reductive group actions on projective varieties.

The decades following the original paper by Guillemin and Sternberg have given rise to a multitude of new research directions. For instance, an infinite-dimensional analogue of the result within Yang-Mills theory (see Atiyah and Bott's article *The Yang-Mills Equations over Riemann Surfaces* [AB83]). Or, in geometric quantisation, where the result can be characterised by the phrase 'quantisation commutes with reduction' (see the book by Guillemin et al. titled *Moment maps, cobordisms, and Hamiltonian group actions*, for a survey [Gui+02]).

We hope that the reader, having read this thesis, is equipped with the tools to begin tackling some of the generalisations or further questions we have posed. We also hope that the reader has acquired the courage needed to travel the paths laid out by some of the research directions mentioned above.

Appendix
A Collection of Results

Theorem. (Hilbert's Basis Theorem). *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian. That is, every ideal of $R[x]$ is finitely-generated.*

Theorem. (Gluing Lemma). *Suppose U and V are both open (or closed) subsets of a topological space X such that $X = U \cup V$, and let $f : X \to Y$ be a function from X to a topological space Y. If the restrictions of f to U and V are continuous, then f is also continuous.*

Theorem. (Universal Property of Topological Quotient). *Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $X/\!\sim$ denote the set of equivalence classes and let $q : X \to X/\!\sim$ be the quotient map. If $f : X \to Z$ is a continuous map that is constant on equivalence classes, there exists a unique continuous map $f : X/\!\sim \to Z$ such that $g = f \circ q$.*

Theorem. (Regular Level Set Theorem). *Suppose that $F : X \to Y$ is a smooth map. Let $c \in Y$ be a regular value such that $F^{-1}(c)$ is non-empty. Then the regular level set $F^{-1}(c)$ is an embedded submanifold of X of dimension $n - m$.*

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