An Interesting Connection between Symplectic Geometry and Geometric Invariant Theory

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Overview

We are interested in quotients of geometric objects (such as manifolds or varieties) by group actions.

1. Brief overview of symplectic geometry and symplectic reduction
2. Brief overview of algebraic geometry and geometric invariant theory (GIT)
3. The Kempf-Ness Theorem: A connection between symplectic geometry and geometric invariant theory. More specifically, a homeomorphism between the symplectic reduction (symplectic quotient) and GIT quotient

What is symplectic geometry?

- Suppose M is an *even dimensional* smooth manifold (think surface)
- A symplectic form ω is an alternating, non-degenerate $(0, 2)$-tensor field on M (think matrix-valued function) that is closed: $d\omega = 0$

Example

The cotangent bundle $T^*\mathbb{R} \cong \mathbb{R}^2$ of $\mathbb{R}$ is a symplectic manifold with symplectic form given by $\omega = dx \wedge dy$. The symplectic form ω can be represented by the anti-symmetric invertible matrix

$$(\omega_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- Symplectic geometry originated in the study of Hamiltonian mechanics in physics. It generalises the phase space of a physical system.

Quotient manifolds: When is the quotient, a manifold?

Recall: A quotient (or orbit space) of a set X by a group G, given a particular action of G on X, is the set of G-orbits, denoted X/G.

Questions: When is M/G a symplectic manifold, or even just smooth?

- Let K be a *compact Lie group* (such as $\mathrm{U}(n)$) acting *smoothly* and *freely* (trivial stabilisers for every point) on a smooth manifold M.

Theorem

The orbits are embedded submanifolds of M that are diffeomorphic to K and moreover, the set of orbits M/K can be given a smooth manifold structure with dimension $\dim M - \dim K$.

For example, consider the action of $K = S^1$ on $M = \mathbb{C} \setminus \{0\}$ by multiplication: $z \cdot w = zw$. Then the orbit space M/K is homeomorphic to $(0, \infty)$ which is a smooth manifold.

Moment map: To define symplectic quotient

As before, let (M, ω) be a *symplectic manifold* with a smooth, free K-action where K is a *compact Lie group*.

Definition

A moment map $\mu : M \to \mathfrak{K}^*$ (where $\mathfrak{K}^*$ denotes the dual vector space of the Lie algebra of K) is a K-equivariant smooth map satisfying:

1. Hamiltonian condition: For $A \in \mathfrak{K}$, let A_M be the infinitesimal generator of A. Then,
$$d\mu^A = \omega(A_M, \cdot)$$
where $\mu^A : M \to \mathbb{R}$ is the smooth function defined by $\mu^A(x) := \mu(x)(A)$ for $x \in M$.

Remark: For the translation group acting on $T^*\mathbb{R}^3$, the moment map is the linear momentum. For the rotation group SO(3) acting on $T^*\mathbb{R}^3$, the moment map is the angular momentum.

Symplectic reduction/quotient: When is the quotient of a symplectic manifold also symplectic?

Again: Given a compact Lie group K acting on a symplectic manifold (M, ω) and a moment map $\mu : M \to \mathfrak{K}^*$ (if it exists), we define the *symplectic reduction* of M by K as

$$M^{\text{red}} := \mu^{-1}(0)/K.$$

Theorem (Marsden-Weinstein-Meyer [1, 2])

Under suitable conditions, $\mu^{-1}(0)$ is a submanifold of M with dimension $\dim M - \dim K$ *and* $\mathbf{M^{red}}$ **is a symplectic manifold** *of dimension* $\dim M - 2\dim K$*, with a unique symplectic form ω^{red} inherited from ω on M.*

Aside: Symplectic reduction has physical significance (reduced phase space).

Important Example: Constructing projective space by symplectic reduction

Recall: The complex projective space $\mathbb{CP}^n$ is the set of 1-dimensional subspaces (lines through the origin) of $\mathbb{C}^{n+1}$.

- Consider the action of the compact Lie group $K = S^1$ on the symplectic manifold $(\mathbb{C}^{n+1}, \operatorname{Im}\langle\cdot,\cdot\rangle)$ given by scalar multiplication.
- Family of moment maps for the above action given by

  $$\mu(z) = \|z\|^2 - C$$

  where $C \in \mathbb{R}$. Interesting when $C = 1$: we have $\mu^{-1}(0) = S^{2n+1}$.
- The *symplectic reduction* of $\mathbb{C}^{n+1}$ by S^1 is then

  $$(\mathbb{C}^{n+1})^{\text{red}} = \mu^{-1}(0)/S^1 = S^{2n+1}/S^1 \cong \mathbb{CP}^n$$

  and so $\mathbb{CP}^n$ is a symplectic manifold (by the Marsden-Weinstein-Meyer theorem).
- We call the unique symplectic form induced on $\mathbb{CP}^n$ by this action the Fubini-Study form ω_{FS}.
- We'll come back to this example later!

Symplectic reduction

Remark

The moment map on a symplectic manifold is not necessarily unique (if it exists), so there may be many choices of a moment map. Also, there may be more than one fixed point of the coadjoint action so overall there could be multiple choices for the symplectic reduction. Even if $\eta \in \mathfrak{g}^*$ *is not fixed by the coadjoint action, we could instead quotient* $\mu^{-1}(\eta)$ *by the stabiliser group of* η *for the coadjoint action and call that the symplectic reduction. Furthermore,* $0 \in \mathfrak{g}^*$ *is always fixed by the coadjoint action so we may consider the symplectic reduction*

$$M_0^{red} = \mu^{-1}(0)/G.$$

What is algebraic geometry? Affine varieties

Algebraic geometry originated in the study of solutions to systems of polynomial equations, and such solution sets are called *algebraic varieties*. From here on, we work over the *algebraically closed* field $\mathbb{C}$. We begin with the affine case.

- An affine variety $X \subseteq \mathbb{C}^n$ is the zero locus (solution set) of a subset S of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.
- The coordinate ring $\mathbb{C}[X]$ is the (finitely-generated) $\mathbb{C}$-algebra of (polynomial) functions from the variety X to $\mathbb{C}$ (like the algebra of smooth functions $C^\infty(M)$ for a smooth manifold M).

Example:

In fact, there is a duality between affine varieties and their coordinate rings...

Algebra-Geometry duality (bijective correspondence)

For a morphism $\varphi : X \to Y$ of affine varieties X and Y (like smooth maps between smooth manifolds), the pullback of φ between their coordinate rings, $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, is a $\mathbb{C}$-algebra homomorphism.
Algebra-Geometry duality:

$$\{\text{morphisms of affine varieties } X \to Y\}$$

$$\updownarrow$$

$$\{\text{morphisms of coordinate rings } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$

Key point: This duality allows us to study affine varieties purely by studying algebraic functions on it.
Remark: A big distinction between the algebra of smooth functions on a smooth manifold and the algebra of functions on an affine variety, is that the latter recovers the geometry of the variety *completely*.

Algebraic geometry: Projective varieties

Now we introduce the projective case.
Recall: A polynomial is *homogeneous* of degree d if

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$$

for every $\lambda \in \mathbb{C}^\times$ and $(x_0, \ldots, x_n) \in \mathbb{C}^n \setminus \{0\}$.

- A projective variety $X \subseteq \mathbb{CP}^n$ contained in the complex projective space $\mathbb{CP}^n$ is the zero locus (solution set) of a subset of *homogeneous* polynomials in $\mathbb{C}[x_0, \ldots, x_n]$.
- Note that $\mathbb{CP}^n$ can be covered by affine patches U_i and a projective variety $X \subseteq \mathbb{CP}^n$ can also be covered by affine patches $X \cap U_i$.
- In other words, affine varieties glue to form projective varieties.

Remark: Algebra-geometry duality doesn't exist here, so projective geometry is much more complicated.

Reductive groups

Remark: To study quotients in algebraic geometry, we need a notion of an "algebraic" group, but it turns out we need an even stronger notion.

A complex reductive group G is a group which has completely reducible finite-dimensional complex representations.

Theorem

There is a one-to-one correspondence between complex reductive groups and compact real Lie groups given by complexification.

Some examples of pairs of complex reductive groups and compact real Lie groups (G, K) include: $(\mathbb{C}^\times, S^1)$, $(\mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n))$, $(\mathrm{SL}_n(\mathbb{C}), \mathrm{SU}(n))$.

Geometric Invariant Theory (GIT): How does one study quotients in algebraic geometry? (affine case)

Idea: Use the *algebra-geometry duality*.

- Let G be a complex reductive group acting on an affine variety X.
- Induces an action of G on the coordinate ring $\mathbb{C}[X]$ by

$$(g \cdot f)(x) := f(g^{-1}x)$$

  for $g \in G$, $f \in \mathbb{C}[X]$, and $x \in X$.
- The *ring of invariants* is the subring

$$\mathbb{C}[X]^G := \{f \in \mathbb{C}[X] \mid g \cdot f = f \text{ for all } g \in G\}.$$

- *(Affine) GIT quotient* defined by affine variety corresponding to ring of invariants:

$$\mathbb{C}[X]^G \rightsquigarrow X /\!\!/ G$$

Note: $X /\!\!/ G$ may differ from the orbit space X/G!

GIT: How does one study quotients in algebraic geometry? (projective case)

Now suppose that G is a complex reductive group acting on a *complex projective variety* $X \subseteq \mathbb{CP}^n$ by a linear representation of G in $\mathrm{GL}_{n+1}(\mathbb{C})$.

Idea: Cover a 'nice' subset $X^{ss} \subseteq X$ of semistable points by certain G-invariant affine subsets and then glue the (affine) GIT quotients of those to form the (projective) GIT quotient $X^{ss} /\!\!/ G$ which is a projective variety.

Question

Is the projective GIT quotient $X^{ss} /\!\!/ G$ the same as the orbit space X/G?

Answer: No in general. But it contains a subvariety X^s/G which is the orbit space of 'stable' points X^s. We'll look at an example in the following slide.

Example: Constructing projective space via GIT

Consider the action of $G = \mathbb{C}^\times$ on $X = \mathbb{C}^{n+1}$ ($n \geq 1$) by

$$\lambda \cdot (z_0, \dots, z_n) := (\lambda z_0, \dots, \lambda z_n).$$

The coordinate ring of X is $\mathbb{C}[X] = \mathbb{C}[z_0, \dots, z_n]$ and the ring of invariants is given by $\mathbb{C}[X]^G = \mathbb{C}$. Thus, the affine GIT quotient is $\mathbb{C}^{n+1} /\!\!/ \mathbb{C}^\times = \text{pt}$.

If we instead consider the orbit space $\mathbb{C}^{n+1}/\mathbb{C}^\times$:

Problem: 'Bad' point causing the issue here is the origin. If we remove this bad point and consider the open set $X^{ss} = \mathbb{C}^{n+1} \setminus \{0\}$, we cannot take the affine GIT quotient since X^{ss} is no longer affine.

Example: Constructing projective space from GIT

Solution: Cover $X^{ss} = \mathbb{C}^{n+1} \setminus \{0\}$ by the affine patches

$$U_i = \{(z_0, \ldots, z_n) \in X^{ss} \mid z_i \neq 0\} \text{ for } i = 0, \ldots, n.$$

Their corresponding coordinate rings are

$$\mathbb{C}[U_i] = \mathbb{C}[z_0, \ldots, z_i^{\pm 1}, \ldots, z_n] \text{ for } i = 0, \ldots, n.$$

Note that the affine patches are all G-invariant so we may take their (affine) GIT quotients by calculating the ring of invariants for each of their coordinate rings. The ring of invariants for each $i = 0, \ldots, n$ is given by

$$\mathbb{C}[U_i]^G = \mathbb{C}\left[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right] \cong \mathbb{C}[z_1^{(i)}, \ldots, z_n^{(i)}]$$

which correspond to the affine varieties $\mathbb{C}^n_{z_1^{(i)}, \ldots, z_n^{(i)}}$. Now taking the disjoint union of these varieties, we obtain the GIT quotient $X^{ss} /\!\!/ G = \mathbb{CP}^n$ which **coincides with the orbit space X^{ss}/G.**

Kempf-Ness Theorem: Connecting the two quotients

Reminder: $\mathbb{CP}^n$ is a symplectic manifold and so a smooth complex projective variety $X \subseteq \mathbb{CP}^n$ is also symplectic by restricting the Fubini-Study symplectic form (from earlier example) on $\mathbb{CP}^n$. Let K be the maximal compact subgroup of G. **Two different quotients:**

$$G \curvearrowright X \text{ (as a complex projective variety)} \implies \text{GIT quotient } X^{ss} /\!\!/ G$$

$K \curvearrowright X$ (as a symplectic manifold) and a moment map $\mu : X \to \mathfrak{K}^*$ exists

$$\implies \text{symplectic reduction } \mu^{-1}(0)/K$$

Theorem (Kempf-Ness [3, 4])

Given the above conditions, there is a homeomorphism

$$\mu^{-1}(0)/K \cong X^{ss} /\!\!/ G$$

between the symplectic reduction and the GIT quotient.

Kempf-Ness Theorem: Example

Example

We have seen that for $X = \mathbb{C}^{n+1}$ with $G = \mathbb{C}^\times$ acting by scalar multiplication, that $X^{ss} = \mathbb{C}^{n+1} \setminus \{0\}$ and $(\mathbb{C}^{n+1} \setminus \{0\}) /\!\!/ \mathbb{C}^\times = \mathbb{CP}^n$. On the other hand, $K = S^1$ is the maximal compact subgroup of G and the symplectic reduction is $\mu^{-1}(0)/K = S^{2n+1}/S^1 \cong \mathbb{CP}^n$.

For $n = 1$ we have the following picture:

The point here is that the G-orbits of 'nice' points intersect the level set $\mu^{-1}(0)$ at *unique* K-orbits.

References I

- Marsden, J., Weinstein, A., 'Reduction of symplectic manifolds with symmetry', *Rep. Mathematical Phys.*, **5** (1974), 121–130.
- Meyer, K., 'Symmetries and integrals in mechanics, Dynamical Systems' (Proc. Sympos., Univ. Bahia, Salvador, 1971), 259-272. Academic Press, New York, 1973.
- G. Kempf and L. Ness, 'The length of vectors in representation spaces', Algebraic Geometry, Lecture Notes in Mathematics, vol. 732, Springer Berlin / Heidelberg, 1979, pp. 233-243.
- D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory, third ed., Springer, 1993.

Questions